UNIT I

NOTES

1. ALGEBRA OF DIFFERENTIAL FUNCTIONS

STRUCTURE

INTRODUCTION

Students have studied the concept of differentiability in lower classes. In the present chapter we shall study chain rule of differentiability, Mean value theorems and their geometrical interpretations.

DERIVABILITY

Derivability at an interior point. *A function f defined on [a, b] is said to be derivable or differentiable at* $c \in (a, b)$ *if* $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ *exists.*

In case the limit exists, the same is called *the derivative of f at c or differential co-efficient of f at c* and is denoted by $f'(c)$ or $\frac{d}{dx} f(x) \Big|_{x=c}$. The process of getting $f'(c)$ is called the *differentiation.*

Left hand derivative

$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$, if exists is called the *left hand derivative* of f at c and is denoted by $f'(c^-)$.

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Right hand derivative

$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$, if exists is called the *right hand derivative* of f at c and is denoted by $f'(c^+)$.

Clearly, $f'(c)$ exists if $f'(c^-)$ and $f(c^+)$ both exist and are equal.

Another Form. A function f has a finite derivative $f'(c)$ at c

iff $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

i.e., iff $$\frac{f(c+h) - f(c)}{h} = f'(c) + \eta \qquad \text{where } \eta \to 0 \text{ as } h \to 0$$

i.e., iff $$f(c+h) = f(c) + hf'(c) + h\eta, \text{ where } \eta \to 0 \text{ as } h \to 0.$$

$h f'(c)$ is called the differential of f at c.

Derivatives at the end points. *A function f defined on [a, b] is said to be derivable at a if* $\lim_{h \to 0^+} \frac{(a+h) - f(a)}{h}$ *exists.*

The above limit is called the derivative of f at a and is denoted by $f'(a)$ (by which we mean $f'(a^+)$).

A function f defined on [a, b], is said to be derivable at b if $\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$ *exists.*

The above limit is called the derivative of f at b and is denoted by $f'(b)$ (by which we mean $f'(b^-)$).

Derivability in an interval. *A function f defined on an interval I is said to be derivable on I, if it is derivable at every point of the interval I.*

Derived function. If a function f defined on an interval I is derivable on the interval I, then there exists another function f' with domain I such that the value of f' for any $x \in$ I is $f'(x)$. The function f' is called the *derived function* of f or the derivative of f.

Theorem. *If a function is derivable at a point, then it is continuous at the point.*

Proof. Let f be derivable at a point a.

Then, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists and is $f'(a)$

Now $$\lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} . h\right]$$
$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} . \lim_{h \to 0} h = f'(a).0 = 0.$$

$\Rightarrow$ $$\lim_{h \to 0} f(a+h) = f(a)$$

$\Rightarrow$ f is continuous at a.

Remark. While continuity of a function is a necessary condition for the derivability of the function, it is not a sufficient condition as is clear from the following examples :

(*i*) Consider the function $f(x) = |x|$, $x \in$ R

By definition $$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Here $f(0) = 0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 \quad \Rightarrow \quad f(0^-) = f(0) = f(0^+)$$

$\Rightarrow$ the function is continuous at $x = 0$

Now $$f'(0^-) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h-0}{h} = -1$$

$$f'(0^+) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h-0}{h} = 1$$

$\Rightarrow$ $f'(0^-) \neq f'(0^+)$

$\Rightarrow$ $f'(0)$ does not exist.

(*ii*) Consider the function $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

It is easy to see that $f(x)$ is continuous at $x = 0$.

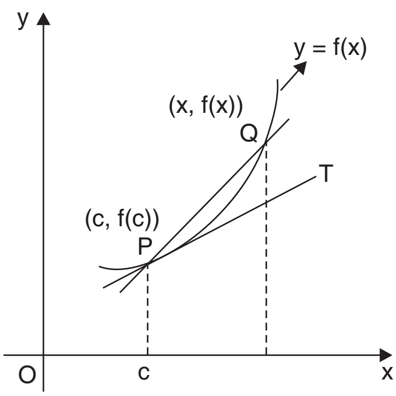
Now $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h}$$

$= \lim_{h \to 0} \sin \dfrac{1}{h}$, which does not exist as $\sin \dfrac{1}{h}$ oscillates between -1 and 1 infinitely many times when $h \to 0$. Hence $f'(0)$ does not exist.

Geometrical Interpretation

Consider the graph of the function $y = f(x)$. Let P(c, $f(c)$) be a point on it.

Then, by definition, $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$, $x \neq c$.



But, $\dfrac{f(x) - f(c)}{x - c}$ gives the slope of secant PQ, where Q(x, $f(x)$). As $x \to c$, the point Q moves along the curve and tends to coincide with P and the secant PQ taking the position PT, tangent at P.

Hence, $f'(c)$ gives the slope of tangent to the curve at P.

Example. *Prove that the function f defined by*

$$f(x) = |x + 1| + |x| \quad \forall\, x \in R$$

is continuous but not derivable at x = – 1, 0.

Sol. (*i*) *Continuity of f at x = – 1* $f(-1) = 0 + 1 = 1$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} [\,|x+1| + |x|\,]$$

$$= \lim_{x \to -1^-} [-(x+1) + (-x)] = 1 \qquad \left[\because\ |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \le 0 \end{cases}\right]$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} [\,|x+1| + |x|\,]$$

$$= \lim_{x \to -1^+} |(x+1) + (-x)| = 1$$

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Thus $\lim_{x \to -1^-} f(x) = f(-1) = \lim_{x \to -1^+} f(x)$

$\Rightarrow$ f is continuous at $x = -1$

Derivability of f at x = – 1

We have $f'(-1^-) = \lim_{h \to 0^-} \dfrac{f(-1+h) - f(-1)}{h}$

$$= \lim_{h \to 0^-} \frac{|-1+h+1| + |-1+h| - 1}{h} \qquad (\because\ |h| = -h \text{ as } h < 0)$$

$$= \lim_{h \to 0^-} \frac{-h+1-h-1}{h} = -2$$

$$f'(-1^+) = \lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0^+} \frac{|-1+h+1| + |-1+h| - 1}{h} = \lim_{h \to 0^+} \frac{h+1-h-1}{h} = 0$$

$\therefore$ $f'(-1^-) \neq f'(-1^+)$

$\Rightarrow$ $f'(-1)$ does not exist

$\Rightarrow$ f is not derivable at $x = -1$.

(*ii*) The case $x = 0$ can be dealt with on the same lines.

DERIVATIVE OF THE COMPOSITE OF TWO FUNCTIONS

Theorem. *(Chain Rule of differentiability)*

Let f be a function defined from a closed interval *I = [a, b] to R. Let g be another function defined from an interval J to R, where f(I) ⊆ J. Let φ be a function defined from I to R by :*

$$\phi(x) = (gof)(x) = g(f(x)),\ \forall\ x \in I.$$

If (i) $x_0 \in (a, b)$ (ii) $f(x_0)$ is an interior point of f(I) (iii) f is differentiable at x_0 and (iv) g is differentiable at $f(x_0)$, then φ is differentiable at x_0 and

$$\phi'(x_0) = (gof)'(x_0) = g'(f(x_0))\, f'(x_0).$$

Proof. Let $y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \lambda(x), \text{ where } \lambda(x) \to 0 \text{ as } x \to x_0.$$

$$\Rightarrow \quad f(x) - f(x_0) = (x - x_0)\,(f'(x_0) + \lambda(x)) \qquad ...(1)$$

Since g is differentiable at y_0, we have

$$\frac{g(y) - g(y_0)}{y - y_0} = g'(y_0) + \mu(y), \text{ where } \mu(y) \to 0 \text{ as } y \to y_0$$

$\Rightarrow \quad g(y) - g(y_0) = (y - y_0)\,[g'(y_0) + \mu(y)]$ ...(2)

Now, $\phi(x) - \phi(x_0) = (gof)(x) - (gof)\,(x_0)$

$= g(f(x)) - g(f(x_0)) = g(y) - g(y_0)$

$= (y - y_0)\,[g'(y_0) + \mu(y)]$ [(By (2)]

$= [f(x) - f(x_0)]\,[g'(y_0) + \mu(y)]$

$= (x - x_0)\,[f'(x_0) + \lambda(x)]\,[g'(y_0) + \mu(y)]$ [By (1)]

Hence, for $x \neq x_0$,

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} = [g'(y_0) + \mu(y)]\,[f'(x_0) + \lambda(x)]$$

f being differentiable at x_0, f is continuous at x_0 also. ...(3)

$\therefore \quad f(x)\,(= y) \to f(x_0)(= y_0)$ as $x \to x_0$

and consequently $\quad \mu(y) \to 0$ as $x \to x_0$

Also $\quad \lambda(x) \to 0$ as $x \to x_0$

$\therefore$ Taking limits as $x \to x_0$ in (3), we get

$$\phi'(x_0) = g'(y_0)\,f'(x_0)$$

or

$$(gof)'\,(x_0) = g'\,(f(x_0))\,f'(x_0).$$

Derivative of the Inverse Function

Theorem. *Let $f : I \to R$ be continuous and one-to-one on an interval I. Let $x_0 \in I$. If f is differentiable at x_0 and $f'(x_0) \neq 0$ then f^{-1} is also differentiable at $f(x_0)$ and*

$$(f^{-1})'\,(f(x_0)) = \frac{1}{f'(x_0)}.$$

Proof. Let $\quad y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \lambda(x), \text{ where } \lambda(x) \to 0 \text{ as } x \to x_0$$

$\Rightarrow \quad f(x) - f(x_0) = (x - x_0)\,(f'(x_0) + \lambda(x))$ ...(1)

Now, $\quad f^{-1}(y) - f^{-1}(y_0) = f^{-1}(f(x)) - f^{-1}(f(x_0)) = x - x_0$ (By def. of f^{-1})

$$\therefore \quad \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}$$

$$= \frac{1}{f'(x_0) + \lambda(x)} \qquad \text{[By (1)]}$$

$$\Rightarrow \quad \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0) + \lambda(x)} \qquad ...(2)$$

Since f is continuous at x_0, it follows that f^{-1} is also continuous at $f(x_0)\,(= y_0)$.

$\therefore \quad f(x) \to f(x_0) \quad \Rightarrow \quad f^{-1}\,(f(x)) \to f^{-1}\,(f(x_0))$

$\therefore \quad y \to y_0 \quad \Rightarrow \quad x \to x_0.$

But, when $x \to x_0$, $\lambda(x) \to 0$

$\therefore$ from (2), $\displaystyle\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$

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$\Rightarrow$ $\displaystyle (f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ or $\displaystyle (f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$.

ROLLE'S THEOREM*

If a function f defined on [a, b] is such that it is

(i) continuous on [a, b] *(ii) derivable on (a, b)*

(iii) f(a) = f(b)

then there exists at least one $c \in (a, b)$ *such that* $f'(c) = 0$.

Proof. If f is a constant function, then $f'(x) = 0 \ \forall \ x \in (a, b)$ and we have nothing to prove. So, let f be not a constant function.

Since f is continuous on $[a, b]$, f is bounded and attains its bounds on $[a, b]$.

Let M = l.u.b. of f and m = g.l.b. of f on $[a, b]$.

Then, $\exists \ c, d \in [a, b]$, such that $f(c) = \text{M}$ and $f(d) = m$. Clearly $\text{M} \neq m$ (as otherwise $f(x) = \text{M} \ \forall \ x$ and hence would be a constant function). Therefore, either of M and m is different from $f(a)$ and $f(b)$. ($\because$ $f(a) = f(b)$)

Suppose that $\text{M} = f(c)$ is different from $f(a)$ and $f(b)$. Then $c \neq a$ and $c \neq b$.

$\Rightarrow$ $c \in (a, b)$.

Since f is derivable on (a, b), f is derivable at c also.

$$\therefore \quad \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

both exist and are equal each being equal to $f'(c)$

Since $f(c) = \text{M}$ (= l.u.b. of f)

$$\therefore \quad \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\Rightarrow \quad f'(c) \geq 0 \quad \text{and} \quad f'(c) \leq 0$$

$$\Rightarrow \quad f'(c) = 0.$$

Geometrical Interpretation of Rolle's Theorem

Let the curve $y = f(x)$ which is continuous on $[a, b]$ and derivable on (a, b) be drawn. Let $\text{A}(a, f(a))$ and $\text{B}(b, f(b))$ be points on the curve corresponding to $x = a$ and $x = b$. Also, $f(a) = f(b)$ *i.e.*, the ordinates of A and B are equal. Then, it is clear from the figures that there is one point $\text{P}(c, f(c))$ on the curve in Fig. (*i*) (and more than one point namely P_1, P_2, P_3, P_4 in Fig. (*ii*)) where tangent is parallel to x-axis. That is, $f'(c) = 0$.

*After the name of a British mathematician Michael Rolle (1652—1719).

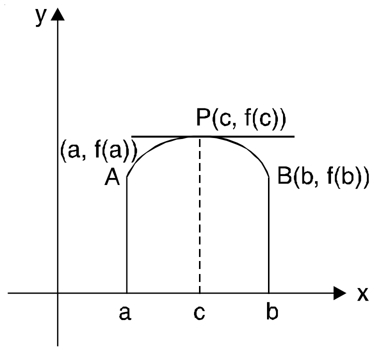

Fig. (*i*)

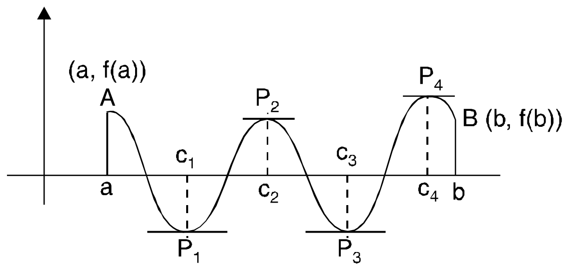

Fig. (*ii*)

Remarks 1. Converse of Rolle's theorem is not true, *i.e.*, $f'(x)$ may vanish at a point $c \in (a, b)$ without $f(x)$ satisfying the three conditions of Rolle's theorem.

For example, the function $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 2, & \frac{1}{2} \le x \le 1 \end{cases}$ satisfies none of the conditions of Rolle's theorem yet $f'(x) = 0$ for each $x \in (0, 1)$.

2. There may exist more than one real number $c \in (a, b)$ such that $f'(c) = 0$ but Rolle's theorem ensures the existence of *at least* one such c.

3. The hypothesis of Rolle's theorem cannot be weakened. Following examples illustrate it

(*i*) Consider $f(x) = \begin{cases} 3x, 0 < x < 1 \\ 4, x = 1 \end{cases}$

Here, $f(x)$ is not continuous at $x = 1$ and $f'(x) \ne 0$ for $x \in (0, 1)$.

(*ii*) Consider $f(x) = |\ x\ |$, $[-1, 1]$.

Then, f is not differentiable at $x = 0 \in (-1, 1)$. $f'(x) \ne 0$ for $x \in (-1, 1) - \{0\}$.

(*iii*) Consider $f(x) = x$, $[1, 2]$.

Here, $f(1) = 1$, $f(2) = 2$ and hence $f(1) \ne f(2)$ but $f'(0) = 1 \ne 0$ for $x \in (1, 2)$.

4. Algebraically, Rolle's theorem means that if $f(x)$ is a polynomial function in x and $x = a$ and $x = b$ are two roots of the equation $f(x) = 0$, then there is, at least one root of the equation $f'(x) = 0$ which lies between a and b.

Another form of Rolle's Theorem

If a function 'f' defined on $[a, a + h]$ is such that it is

(*i*) continuous on $[a, a + h]$ (*ii*) derivable on $(a, a + h)$

(*iii*) $f(a) = f(a + h)$.

Then there exists at least one real number θ, $0 < \theta < 1$ such that $f'(a + \theta h) = 0$

[Hint. Take $b = a + h$ and $c = a + \theta h$ in Rolle's theorem.]

To test the continuity of $f(x)$ in a closed interval $[a, b]$

To test the continuity of f on a closed interval $[a, b]$, we check three things :

(*i*) *$f(x)$ is not infinite on the closed interval* $[a, b]$.

For example, $f(x) = \dfrac{3x^2 + 5}{(x - 1)}$ *is infinite at $x = 1$. Hence f is not continuous on any closed interval containing 1 i.e.,* $[0, 1]$, $[0, 2]$, $[1, 3]$.

(*ii*) *$f(x)$ is not imaginary on closed interval* $[a, b]$.

For example, $f(x) = \sqrt{x-1}$ will be imaginary when $x < 1$. Hence $f(x)$ will be discontinuous on any interval containing $x < 1$.

(*iii*) *$f(x)$ has no break on the closed interval* $[a, b]$.

For example, $f(x) = x^2 + 3$ for $x \le 1 = 5x$ for $x > 1$,

has a break at $x = 1$, so it may not be continuous at $x = 1$. At such points, we have to check the continuity. The above function is continuous on the intervals of the type $[a, 1]$, but not continuous on the interval containing 1 and of the type $[1, b]$.

Alternatively, we may proceed as :

(*i*) If $f(x)$ is a polynomial function of x, then it is continuous for all real x as *polynomial functions are* **always** *continuous for real x*.

(*ii*) If $f(x)$ is not a polynomial function, then we find $f'(x)$. If $f'(x)$ is finite, definite and real in $[a, b]$, then $f(x)$ is derivation $[a, b]$ and hence continuous for all x on $[a, b]$.

To check the derivability of $f(x)$ in (a, b), we may proceed as :

(*i*) If $f(x)$ is a polynomial function of x, then $f(x)$ is derivable on (a, b) as a polynomial function is always derivable for all real values of x.

(*ii*) If $f(x)$ is not a polynomial, then find $f'(x)$ and if $f'(x)$ is finite, definite and real on (a, b), then f is derivable on (a, b).

Example 1. *Verify Rolle's theorem to the function*

$$f(x) = e^{-x} \sin x,\ [0, \pi].$$

Sol. Here $\quad f(x) = e^{-x} \sin x,\ \forall\ x \in [0, \pi]$

e^{-x} and $\sin x$ are both continuous as well as derivable $\forall\ x \in \mathrm{R}$.

$\Rightarrow\quad e^{-x} \sin x$ is continuous as well as derivable $\forall\ x \in \mathrm{R}$.

$\Rightarrow\quad e^{-x} \sin x$ is continuous as well as derivable on $[0, \pi]$ also.

Also $\quad f(0) = e^0 \sin 0 = 0,\ f(\pi) = e^{-\pi} \sin \pi = 0$

$\therefore\quad f(0) = f(\pi)$

Thus f satisfies all the 3 conditions of Rolle's theorem on $[0, \pi]$. Hence Rolle's theorem is applicable on $[0, \pi]$.

$\therefore\quad \exists\ c \in (0, \pi)$ such that $\quad f'(c) = 0$.

But $\quad f'(x) = -e^{-x} \sin x + e^{-x} \cos x$

$\therefore\quad f'(c) = 0 \quad\Rightarrow\quad -e^{-c} \sin c + e^{-c} \cos c = 0$

$\Rightarrow\quad e^{-c} (-\sin c + \cos c) = 0$

$\Rightarrow\quad -\sin c + \cos c = 0 \qquad (\because\ e^{-c} \ne 0)$

$\Rightarrow\quad \tan c = 1 \quad\Rightarrow\quad c = \dfrac{\pi}{4} \in (0, \pi).$

Hence Rolle's theorem is verified to f on $[0, \pi]$.

Example 2. *Examine the applicability of Rolle's theorem to the function*

$$f(x) = 2 + (x-1)^{2/3} \text{ on } [0, 2].$$

Sol. $f'(x) = \dfrac{2}{3(x-1)^{1/3}}$, which is defined for every x except at $x = 1$. So, the function is derivable on $(0, 2)$ except perhaps at $x = 1$.

Also, $f(0) = f(0^+)$ and $f(2^-) = f(2)$.

Since derivability implies continuity, therefore the function is continuous on [0, 2] except possibly at $x = 1$.

Now $| f(x) - f(1) | = | (x - 1)^{2/3} | < \varepsilon$ for $| x - 1 | < \varepsilon^{3/2}$

i.e., for any $\varepsilon > 0$, $\exists\ \delta\ (= \varepsilon^{3/2})$ such that

$$| f(x) - f(1) | < \varepsilon, \text{ whenever } | x - 1 | < \delta.$$

$\Rightarrow$ f is continuous at $x = 1$ also.

Let us now check the derivability of f at $x = 1$.

$$f'(1^-) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2 + (1 + h - 1)^{2/3} - 2}{h}$$

$$= \lim_{h \to 0^-} \frac{h^{2/3}}{h} = \lim_{h \to 0^-} \frac{1}{h^{1/3}} = -\infty$$

$$f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{1}{h^{1/3}} = \infty$$

$\Rightarrow$ $f'(1^-) \neq f'(1^+)$ $\Rightarrow$ $f'(1)$ does not exist

$\Rightarrow$ f is not derivable on (0, 2)

$\Rightarrow$ Rolle's theorem is not applicable on [0, 2].

Example 3. *Show that for no value of k, the equation* $x^3 - 3x + k = 0$ *has two different roots in (0, 1).*

Sol. Let $f(x) = x^3 - 3x + k$

Let, if possible, the equation $f(x) = 0$ has two different roots α, β in (0, 1), $f(x)$ being a polynomial, f is continuous and derivable $\forall\ x \in R$.

$\Rightarrow$ f is continuous on $[\alpha, \beta]$ and f is derivable on (α, β).

Also $f(\alpha) = f(\beta) = 0$

$\therefore$ by Rolle's theorem, $\exists\ c \in (\alpha, \beta)$ such that

$$f'(c) = 0 \quad \Rightarrow \quad 3c^2 - 3 = 0 \qquad (\because \ f'(x) = 3x^2 - 3)$$

$\Rightarrow$ $c = \pm 1$ which contradicts that $c \in (\alpha, \beta) \subseteq (0, 1)$.

Hence the result.

Example 4. *Prove that if* $a_0, a_1, \ldots\ldots, a_n, \in R$ *are such that*

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0,$$

then there exists at least one $x \in (0, 1)$ *such that* $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$.

Sol. Consider the function

$$f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + \ldots + a_{n-1} \frac{x^2}{2} + a_n x$$

Then, $f(x)$ being a polynomial, f is continuous and derivable $\forall\ x$.

$\Rightarrow$ f is continuous on [0, 1] and derivable on (0, 1). Also

$$f(0) = 0, \ f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0 \text{ (given)}$$

$\therefore$ by Rolle's theorem, $\exists$ at least one $c \in (0, 1)$ such that $f'(c) = 0$.

Now $$f'(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}\,x + a_n$$

$\therefore$ $f'(c) = 0$ for at least one $c \in (0, 1)$

$\Rightarrow$ $a_0x^n + a_1x^{n-1} + ... + a_n$ vanishes at least once in (0, 1).

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Example 5. *Using Rolle's theorem, show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.*

Sol. Let α and β be any two different roots of $e^x \cos x = 1$.

$$\therefore \quad e^{\alpha} \cos\alpha - 1 = 0 \quad \text{and} \quad e^{\beta} \cos\beta - 1 = 0$$

$$\Rightarrow \quad \cos\alpha = e^{-\alpha} \quad \text{and} \quad \cos\beta = e^{-\beta} \quad ...(1)$$

Consider the function f defined by

$$f(x) = e^{-x} - \cos x \text{ on } [\alpha, \beta].$$

$f(x)$ is clearly continuous on $[\alpha, \beta]$ and differentiable on (α, β).

Also, $$f(\alpha) = e^{-\alpha} - \cos\alpha = 0$$

$$f(\beta) = e^{-\beta} - \cos\beta = 0 \quad \text{[By (1)]}$$

$\therefore$ f satisfies all the conditions of Rolle's theorem on $[\alpha, \beta]$.

$\therefore$ There exists c, $\alpha < c < \beta$ such that $f'(c) = 0$

$\Rightarrow$ $-e^{-c} + \sin c = 0$ $\Rightarrow$ $e^c \sin c - 1 = 0$ and $c \in (\alpha, \beta)$

$\Rightarrow$ the equation $e^x \sin x - 1 = 0$ has at least one root c between any two roots of $e^x \cos x - 1 = 0$.

Example 6. *If $f(x)$ is a polynomial, then show that between any two consecutive roots of $f'(x) = 0$, there lies at the most one root of $f(x) = 0$.*

Sol. Let a, b be two consecutive roots of $f'(x) = 0$ such that $a < b$.

Suppose that $f(x) = 0$ has two different roots α, β such that α, β lie between a and b.

$$\therefore \quad f(\alpha) = f(\beta) = 0$$

Also $f(x)$ being a polynomial is continuous on $[\alpha, \beta]$ and derivable on (α, β).

Thus f satisfies all the conditions of Rolle's theorem.

$\therefore$ There exists at least one $c \in (\alpha, \beta)$ such that $f'(c) = 0$

$\Rightarrow$ c is a root of $f'(x) = 0$

$\therefore$ Between two consecutive roots a and b of $f'(x) = 0$, there lies a root c of $f'(x) = 0$ [$\because$ *c lies between a and b*]

which is a contradiction. Hence our supposition is wrong.

$\therefore$ Between any two consecutive roots of $f'(x) = 0$, there lies at most one root of $f(x) = 0$.

EXERCISE 1

1. Verify Rolle's theorem for the following functions :
   (*i*) $f(x) = (x - a)^m (x - b)^n$ on $[a, b]$ where $m, n \in \mathbb{N}$
   (*ii*) $f(x) = x^2 - 6x + 8$ on [2, 4].
2. Discuss the applicability of Rolle's theorem to the function $f(x) = |\, x \,|$ on [– 1, 1].
3. Verify Rolle's theorem for the function $f(x) = \begin{cases} x^2 + 1, & 0 \le x \le 1 \\ 3 - x, & 1 \le x \le 2 \end{cases}$.

4. If $f(x)$ is a polynomial in x and $f'(x) \neq 0$ anywhere between a and b, show that $f(x)$ can have at the most one root in $[a, b]$.

[**Sol.** If possible, let α, β be two different roots of $f(x) = 0$ in $[a, b]$. $f(x)$ being a polynomial in x, is continuous on $[\alpha, \beta]$ and differentiable on (α, β). Moreover $f(\alpha) = f(\beta) = 0$.

$\therefore$ by Rolle's theorem, $\exists$ at least one $c \in (\alpha, \beta)$ such that $f'(c) = 0$, which is a contradiction.]

5. If the functions f, ϕ, ψ are continuous on $[a, b]$ and derivable on (a, b), show that there exists a point $\xi \in (a, b)$ such that

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(\xi) & \phi'(\xi) & \psi'(\xi) \end{vmatrix} = 0$$

[**Hint.** Let
$$F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}$$

F is continuous on $[a, b]$ and differentiable on (a, b). Also $F(a) = F(b) = 0$.

$\therefore$ By Rolle's theorem, $F'(\xi) = 0$ for $\xi \in (a, b)$ and the result follows.]

6. If $f'(x)$ and $g'(x)$ exist for all $x \in [a, b]$ and $g'(x) \neq 0 \ \forall \ x \in [a, b]$, then prove that there exists some c, $a < c < b$ such that $\dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$.

[**Hint.** Apply Rolle's theorem to the function $\phi(x) = f(x).\ g(x) - g(x)\ f(a) - f(x)\ g(b)$.]

7. If f and g are two functions continuous on $[a, b]$ and derivable on (a, b), show that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}.$$

[**Hint.** Apply Rolle's theorem to the function

$$\phi(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} - \frac{x - a}{b - a} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} \text{ on } [a, b]. \Bigg]$$

8. If a function is such that its derivative f', is continuous on $[a, b]$ and derivable on (a, b), then show that there exists a number c between a and b such that

$$f(b) = f(a) + (b - a) f'(a) + \frac{1}{2} (b - a)^2 f''(c).$$

[**Hint.** Consider $\phi(x) = f(x) + (b - x) f'(x) + (b - x)^2 A$, where A is such that $\phi(a) = \phi(b)$

Apply Rolle's theorem to ϕ and get $A = \dfrac{1}{2} f''(x)$.]

9. If a function is twice derivable on $[a, a + h]$, then show that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a + \theta h), \qquad 0 < \theta < 1.$$

10. Show that there is no real number k, for which the equation $x^2 - 3x + k = 0$ has two different roots on $[0, 1]$.

11. If f'' is continuous on $[a, b]$, show that

$$f(c) = \left(\frac{b - c}{b - a}\right) f(a) + \left(\frac{c - a}{b - a}\right) f(b) + \frac{1}{2} (c - a)(c - b) f''(\xi) \text{ where } c \text{ and } \xi \text{ both lie on } [a, b].$$

[**Hint.** Consider the function $\phi(x) = f(x) - \dfrac{b - x}{b - a} f(a) - \dfrac{x - a}{b - a} f(b) - \dfrac{1}{2} (x - a)(b - x) A$, ...(1)

where A is a constant such that $\phi(c) = 0$.

Apply Rolle's theorem to the function ϕ on the intervals $[a, c]$ and $[c, b]$.

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Then, $\phi'(p) = 0,\ \phi'(q) = 0$ for $a < p < c,\ c < q < b$.

and $$\phi'(x) = f'(x) + \frac{f(a) - f(b)}{b - a} + \frac{1}{2}(2x - a - b)\,A$$

Now, apply Rolle's theorem to $\phi'(x)$ on $[p, q]$.

$\therefore$ There exists $\xi \in (p, q)$ such that $f''(\xi) = 0$

$\Rightarrow$ $$0 = f''(\xi) + A \quad \Rightarrow \quad A = -f''(\xi)$$

Using this value of A in (1), we get

$$\phi(x) = f(x) - \frac{f - x}{b - a} f(a) - \frac{x - a}{b - a} f(b) + \frac{1}{2}(x - a)(x - b) f''(\xi)$$

Since $\phi(c) = 0$, we have

$$f(c) = \frac{b - c}{b - a} f(a) - \frac{c - a}{b - a} f(b) + \frac{1}{2}(c - a)(c - b) f''(\xi),$$

where $c, \xi \in [a, b]$ $(\because\ a < p < c,\ c < q < b$ and $\xi \in (p, q)).]$

LAGRANGE'S MEAN VALUE THEOREM* (OR FIRST MEAN VALUE THEOREM OF DIFFERENTIAL CALCULUS)

If a function f defined on [a, b] is

(*i*) *continuous on [a, b] and* (*ii*) *derivable on (a, b)*

then there exists at least one real number c, a < c < b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Consider a function $\phi(x) = f(x) + Ax$

where A is a constant to be determined such that $\phi(a) = \phi(b)$. This gives us

$$A = -\frac{f(b) - f(a)}{b - a} \qquad ...(1)$$

Now, (*i*) Since f is continuous on $[a, b]$ and Ax is continuous on $[a, b]$, therefore, ϕ is continuous on $[a, b]$.

(*ii*) Since f is derivable on (a, b) and Ax is derivable on (a, b), therefore ϕ is derivable on (a, b).

(*iii*) $\phi(a) = \phi(b)$ (By def. of ϕ).

Thus, ϕ satisfies all the conditions of Rolle's theorem on $[a, b]$.

Therefore, there exists a real number c, $a < c < b$,

such that $\phi'(c) = 0$

But, $\phi'(x) = f'(x) + A$

$\therefore$ $$0 = \phi'(c) = f'(c) + A \Rightarrow f'(c) = -A \qquad ...(2)$$

$\therefore$ from (1) and (2) $$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{or} \quad f(b) - f(a) = (b - a) f'(c)$$

Another form of statement

If in the statement of the theorem, b is replaced by a + h, then the number c between a and b may be written as a + θh, 0 < θ < 1.

*After the name of French Mathematician Joseph Louis lagrange (1736–1813).

$$f(a + h) - f(a) = hf'(a + \theta h)$$

or $$f(a + h) = f(a) + hf'(a + \theta h),\ 0 < \theta < 1$$

Cor. 1. *If f is continuous on [a, b] and is differentiable on (a, b), then*

(*i*) $f'(x) = 0\ \forall\ x \in (a, b) \Rightarrow f$ is constant on $[a, b]$.

(*ii*) $f'(x) > 0\ \forall\ x \in (a, b) \Rightarrow f$ is strictly increasing on $[a, b]$.

(*iii*) $f'(x) < 0\ \forall\ x \in (a, b) \Rightarrow f$ is strictly decreasing on $[a, b]$.

Proof (*i*). Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$. Since f is continuous on $[a, b]$, f is continuous on $[x_1, x_2]$. Since f is differentiable on (a, b), f is differentiable on (x_1, x_2).

Therefore, by Lagrange's mean value theorem, there exists at least one c, $x_1 < c < x_2$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

But $f'(c) = 0$ $\quad [\because\ c \in (x_1, x_2) \subset (a, b)$ and $f'(x) = 0\ \forall\ x \in (a, b)]$

$\therefore \quad f(x_2) - f(x_1) = 0$

$\Rightarrow \quad f(x_2) = f(x_1),\ \forall\ x_1, x_2 \in [a, b]$

Thus, the function has the same value and is therefore constant on $[a, b]$.

(*ii*) Let x_1, x_2 (where $x_1 < x_2$) be any two different points of $[a, b]$. Then, proceeding as in case (*i*) above, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \text{ for some } c,\ x_1 < c < x_2.$$

But $f'(c) > 0$ $\quad [\because\ f'(x) > 0,\ \forall\ x \in (a, b)]$

$\therefore \quad f(x_2) - f(x_1) > 0 \quad \Rightarrow \quad f(x_2) > f(x_1).$

Thus, $\quad x_2 > x_1 \quad \Rightarrow \quad f(x_2) > f(x_1),\ \forall\ x_1, x_2 \in [a, b]$

$\therefore$ f is strictly increasing on $[a, b]$.

Remark. For a strictly increasing function f, the derivative f ′(x) need not be strictly positive. For example, consider $f(x) = x^3$, $x \in (-1, 1)$. This function f is strictly increasing. But $f'(x) = 3x^2$ which is zero at $x = 0 \in (-1, 1)$.

(*iii*) Let x_1, x_2 (where $x_1 < x_2$) be any two different points of $[a, b]$.

$$\therefore \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c),\ x_1 < c < x_2$$

$\Rightarrow \quad f(x_2) - f(x_1) < 0 \quad (\because\ f'(c) < 0$ as $f'(x) < 0\ \forall n \in (a, b))$

$\therefore \quad x_1 < x_2 \quad \Rightarrow \quad f(x_2) < f(x_1)$

$\therefore$ f is strictly decreasing on $[a, b]$.

Cor. 2. *If two functions f and g are*

(*i*) *continuous on [a, b]* (*ii*) *derivable on (a, b)*

(*iii*) $f'(x) = g'(x)\ \forall\ x \in (a, b)$,

then f – g is a constant function.

Proof. Let $\quad F(x) = f(x) - g(x),\ \forall\ x \in [a, b]$

Since f and g are continuous on $[a, b]$

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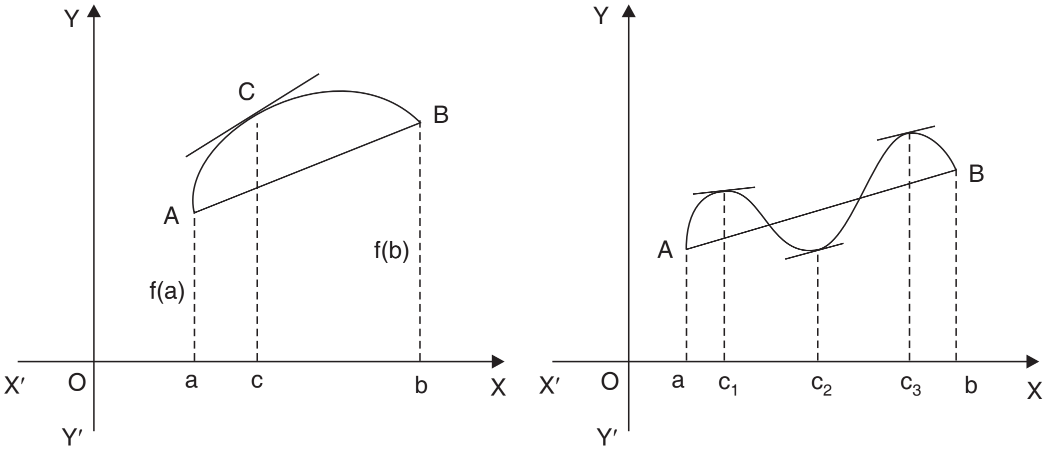
$\therefore$ F is continuous on $[a, b]$

Similarly F is differentiable on (a, b)

Also, $$F'(x) = f'(x) - g'(x) = 0, \ \forall \ x \in (a, b).$$

$\Rightarrow$ F is a constant function on $[a, b]$ *i.e.*, $f - g$ is a constant function on $[a, b]$.

Geometrical Interpretation. Geometrically, Lagrange's Mean Value Theorem says that between two points A and B on the graph of f, there exists at least one point C where the tangent is parallel to the chord AB.



Remark. The hypothesis of Lagrange's Mean Value Theorem cannot be weakened, as is clear from the following examples :

(*i*) Consider the function f defined on [1, 2] as follows :

$$f(x) = \begin{cases} 2, & x = 1 \\ x^2, & 1 < x < 2 \\ 1, & x = 2 \end{cases}$$

Then, f is continuous as well as derivable on (1, 2) but is not continuous at 1 and 2. Thus, first condition is violated ?

Also, $$\frac{f(2) - f(1)}{2 - 1} = -1$$

$$f'(x) = 2x, \ 1 < x < 2$$

$\Rightarrow$ $$f'(x) > 0 \text{ for } 1 < x < 2$$

Thus, $$\frac{f(2) - f(1)}{2 - 1} \neq f'(x) \text{ for any } x \in (1, 2).$$

(*ii*) Consider the function $f(x) = |\, x \,|$ defined on $[-1, 2]$.

Then, f is continuous on $[-1, 2]$ and derivable at all points of $(-1, 2)$ except at $x = 0$. (Thus, second condition is violated).

Now, $$f'(x) = \begin{cases} -1, & \text{if } x \in (-1, 0) \\ 1, & \text{if } x \in (0, 2) \end{cases}$$

Also $$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{1}{3}$$

Thus, $$\frac{f(2) - f(-1)}{2 - (-1)} \neq f'(x) \text{ for any } x \in (-1, 2).$$

CAUCHY'S MEAN VALUE THEOREM

(Second Mean Value Theorem)

If two functions f, g defined on [a, b] are

(*i*) *Continuous on [a, b]* (*ii*) *derivable on (a, b)*

(*iii*) $g'(x) \neq 0$ *for any* $x \in (a, b)$

then, there exists at least one real number $c \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Notice that $g(a) \neq g(b)$ because otherwise, in view of conditions (*i*) and (*ii*), g will satisfy all the conditions of Rolle's theorem and hence $g'(c) = 0$ for some $c \in (a, b)$ contrary to (*iii*) Consider the function ϕ defined by

$$\phi(x) = f(x) + A\, g(x),\ x \in [a, b] \quad ...(1)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

$$\therefore \quad A = -\frac{f(b) - f(a)}{g(b) - g(a)} \quad ...(2)$$

Now, the function ϕ, being the sum of two continuous and derivable functions, is itself

(*i*) continuous on $[a, b]$,

(*ii*) derivable on (a, b)

and (*iii*) $\phi(a) = \phi(b)$ (by definition of ϕ)

$\therefore$ ϕ satisfies all the three conditions of Rolle's theorem. Hence, there exists at least one point $c \in (a, b)$ such that $\phi'(c) = 0$.

But, $\phi'(x) = f'(x) + Ag'(x)$ [From (1)]

$\Rightarrow$ $0 = f'(c) + A\, g'(c)$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{[By (2)]}$$

Another form of statement

If two functions f, g defined on [a, a + h] are (i) continuous on [a, a + h], (ii) derivable on (a, a + h) and (iii) $g'(x) \neq 0$ *for any* $x \in (a, a + h)$, *then there exists at least one real number* θ, $0 < \theta < 1$ *such that*

$$\frac{f(a + h) - f(a)}{g(a + h) - g(a)} = \frac{f'(a + \theta h)}{g'(a + \theta h)}.$$

Note. (*i*) *Lagrange's Mean Value Theorem can be deduced from Cauchy 's Mean Value Theorem by taking* $g(x) = x$.

(*ii*) *Cauchy's Mean Value Theorem cannot be deduced by applying Lagrange's Mean Value Theorem separately to f and g because then, we get*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)},$$ *where* c_1 *and* c_2 *may not be equal.*

Physical Interpretation of Cauchy's Mean Value Theorem

$\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ may be rewritten as $\dfrac{\dfrac{f(b) - f(a)}{b - a}}{\dfrac{g(b) - g(a)}{b - a}} = \dfrac{f'(c)}{g'(c)}$

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which shows that the mean rates of increase of two functions f and g in an interval is equal to the ratio of the actual rates of increases of the two functions at some point within the interval.

Another form of Cauchy's Mean Value Theorem

If two functions f and g defined on $[a, a+h]$ are

(*i*) continuous on $[a, a+h]$

(*ii*) derivable on $(a, a+h)$

(*iii*) $g'(x) \neq 0$ for any $x \in (a, a+h)$ then there exists at least one real number θ, $0 < \theta < 1$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}.$$

Example 1. *Show that if $x > 0$, $\log(1+x) > \dfrac{x}{1+x}$. Also show that $\dfrac{\log(1+x)}{x}$ monotonically decreases as x increases from 0 to ∞.*

Sol. Let $$f(x) = \log(1+x) - \frac{x}{1+x}.$$

$$\therefore \quad f'(x) = \frac{1}{1+x} - \frac{(1+x).1 - x.1}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2} > 0 \qquad (\because x > 0)$$

$\therefore$ f is monotonically increasing when $x > 0$.

$\therefore$ for $x > 0$, $f(x) > f(0)$

$$\Rightarrow \quad \log(1+x) - \frac{x}{1+x} > \log 1 \quad \Rightarrow \quad \log(1+x) - \frac{x}{1+x} > 0$$

$$\Rightarrow \quad \log(1+x) > \frac{x}{1+x} \qquad ...(1)$$

Let $$F(x) = \frac{\log(1+x)}{x}$$

$$\therefore \quad F'(x) = \frac{x \cdot \dfrac{1}{1+x} - \log(1+x)}{x^2}$$

$\therefore$ $F'(x) < 0$ for $x > 0$ [By (1)]

$\therefore$ the function F is monotonically decreasing for $x > 0$.

Example 2. *If f is a twice differentiable function on $[a, b]$ such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$, prove that there is at least one ξ between a and b for which $f''(\xi) < 0$.*

Sol. Since f is twice differentiable on $[a, b]$, therefore f and f' are both continuous and differentiable on $[a, b]$ and hence, in particular, continuous and differentiable on $[a, c]$ and $[c, b]$ for $a < c < b$.

Applying Lagrange's mean value theorem to f on $[a, c]$ and $[c, b]$, we have

$$\frac{f(c)-f(a)}{c-a} = f'(c_1), \quad a < c_1 < c \qquad ...(1)$$

and $$\frac{f(b)-f(c)}{b-c} = f'(c_2),\ c < c_2 < b \qquad \ldots(2)$$

But $$f(a) = f(b) = 0 \qquad \text{(given)}$$

∴ From (1) and (2), $$f'(c_1) = \frac{f(c)}{c-a} \text{ and } f'(c_2) = -\frac{f(c)}{b-c} \qquad \ldots(3)$$

Applying Lagrange's mean value theorem to f' on $[c_1, c_2]$, we have

$$\frac{f'(c_2)-f'(c_1)}{c_2-c_1} = f''(\xi), \quad \text{where } c_1 < \xi < c_2.$$

But, from (3), $f'(c_1) > 0$ and $f'(c_2) < 0$ ($\because$ $f(c) > 0$ for $a < c < b$) (given)

∴ $f''(\xi) < 0$, for at least one ξ between a and b.

Example 3. *Using Lagrange's mean value theorem, show that*

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2} \text{ if } 0 < u < v.$$

Also, deduce that $$\frac{\pi}{4} + \frac{4}{41} < \tan^{-1}\frac{5}{4} < \frac{\pi}{4} + \frac{1}{8}.$$

Sol. Let $$f(x) = \tan^{-1} x,\ u < x < v.$$

$$\Rightarrow \quad f'(x) = \frac{1}{1+x^2}$$

∴ applying Lagrange's Mean Value Theorem to f, we get

$$\frac{\tan^{-1} v - \tan^{-1} u}{v-u} = \frac{1}{1+c^2},\ u < c < v.$$

But, $c > u \Rightarrow \frac{1}{1+c^2} < \frac{1}{1+u^2}$

and $c < v \Rightarrow \frac{1}{1+c^2} < \frac{1}{1+v^2}$

$$\therefore \quad \frac{1}{1+v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v-u} < \frac{1}{1+u^2}$$

$$\Rightarrow \quad \frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}.$$

For the second result, put $v = \frac{5}{4}$ and $u = 1$.

Example 4. *Use Lagrange's mean value theorem to prove that*

$$1 + x < e^x < 1 + xe^x,\ \forall\ x > 0.$$

Sol. Consider the function $\phi(t) = e^t$.

Then, ϕ is continuous on $[0, x]$ and derivable on $(0, x)$. ∴ ϕ satisfies all the conditions of Lagrange's Mean Value Theorem.

∴ there exists a point $c \in (0, x)$ such that

$$\frac{\phi(x)-\phi(0)}{x-0} = \phi'(c)$$

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$\Rightarrow \quad \dfrac{e^x - 1}{x - 0} = e^c \quad \Rightarrow \quad e^x - 1 = xe^c \quad ...(1)$

But, $\quad 0 < c < x \quad \Rightarrow \quad e^0 < e^c < e^x$

$\Rightarrow \quad 1 < e^c < e^x \quad \Rightarrow \quad x < xe^c < xe^x$

$\Rightarrow \quad x < e^x - 1 < xe^x$ [By (1)]

$\therefore \quad 1 + x < e^x < 1 + xe^x.\ \forall\ x > 0.$

Example 5. *Show that* $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ *,* $0 < x < \dfrac{\pi}{2}$*.*

Sol. Now, $\quad \dfrac{\tan x}{x} > \dfrac{x}{\sin x}, \ 0 < x < \dfrac{\pi}{2}$

if $\quad \dfrac{\sin x \tan x - x^2}{x \sin x} > 0, \ 0 < x < \dfrac{\pi}{2}$.

Since $x \sin x > 0$ for $0 < x < \dfrac{\pi}{2}$, it is enough to prove that $\sin x \tan x - x^2 > 0$.

Let $\quad f(x) = \sin x \tan x - x^2, 0 < x < \dfrac{\pi}{2}$.

$\Rightarrow \quad f'(x) = \cos x \tan x + \sin x \sec^2 x - 2x$

$= \sin x + \sin x \sec^2 x - 2x$

Sign of $f'(x)$ cannot be determined because of the presence of $2x$.

But, $\quad f''(x) = \cos x + \cos x \sec^2 x + 2 \sin x \sec^2 x \tan x - 2$

$= (\sqrt{\sec x} - \sqrt{\cos x})^2 + 2 \tan^2 x \sec x > 0, \ 0 < x < \dfrac{\pi}{2}$.

$\therefore \quad f'(x)$ is increasing.

Also, $f'(0) = 0$. Therefore, $f'(x) > 0$ for $0 < x < \dfrac{\pi}{2}$.

Again, since $f'(x) > 0$, $f(x)$ is increasing and because $f(0) = 0$, therefore, $f(x) > 0$, for $0 < x < \dfrac{\pi}{2}$.

Hence, $\quad \dfrac{\tan x}{x} > \dfrac{x}{\sin x}, \ 0 < x < \dfrac{\pi}{2}$.

Remark. The above inequality can also be written as : $\cos x < \left(\dfrac{\sin x}{x}\right)^2, \ 0 < x < \dfrac{\pi}{2}$.

Example 6. *Show that* $\dfrac{\sin a - \sin b}{\cos b - \cos a} = \cot c, \ 0 < a < c < b < \dfrac{\pi}{2}$.

Sol. Let $\quad f(x) = \sin x$ and $g(x) = \cos x, \ x \in (a, b)$.

$\Rightarrow \quad f'(x) = \cos x$ and $g'(x) = -\sin x$.

Since f and g are both continuous as well as differentiable, therefore by Cauchy's Mean Value Theorem on $[a, b]$, we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \ c \in (a, b)$$

$\Rightarrow \quad \dfrac{\sin b - \sin a}{\cos b - \cos a} = \dfrac{\cos c}{-\sin c} \quad \Rightarrow \quad \dfrac{\sin a - \sin b}{\cos b - \cos a} = \cot c, \ a < c < b.$

Example 7. *If* $f(x) = \frac{1}{x^2}$ *,* $g(x) = \frac{1}{x}$ *, then show that in Cauchy's mean value theorem, c is the harmonic mean between a and b.*

Sol.
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} = \frac{-2c^{-3}}{-c^{-2}}$$

$$\Rightarrow \frac{a+b}{ab} = \frac{2}{c} \Rightarrow c = \frac{2ab}{a+b}.$$

EXERCISE 2

1. Examine the validity of the hypothesis and the conclusion of Lagrange's Mean Value Theorem for the following functions :

   (*i*) $f(x) = |x|, x \in [-1, 1]$ (*ii*) $f(x) = x(x-1)(x-2), x \in [0, \frac{1}{2}]$

   (*iii*) $f(x) = x^{1/3}, x \in [-1, 1]$ (*iv*) $f(x) = 2x^2 - 7x + 10, x \in [2, 5]$.

2. Use Lagrange's Mean Value Theorem to prove that $|\sin x - \sin y| \le |x - y|, \forall x, y \in$ R.

   [**Hint.** Consider $\phi(t) = \sin t, t \in [x, y]$.

   Apply Lagrange's Mean Value Theorem to $\phi(t)$ and use $|\phi'(t)| = |\cos t| \le 1$.]

3. If $a = -1$, $b \ge 1$, and $f(x) = \frac{1}{|x|}$, show that the conclusion of Lagrange's mean value theorem are not satisfied on the interval $[a, b]$ but the conclusion of the theorem is true iff $b > 1 + \sqrt{2}$.

   [**Hint.** $f(0)$ is not defined, so let $f(0) = \lambda$, some finite quantity. Since $f'(0^+) \to -\infty$ and $f'(0^-) \to \infty$, f is not differentiable at $x = 0 \in (a, b)$. So, conditions of mean value theorem are not satisfied.

   But $$\frac{f(b)-f(a)}{b-a} = f'(c), \quad a < c < b$$

   $$\Rightarrow \frac{1}{|b|} - \frac{1}{|a|} = (b-a)\left\{\frac{d}{dx}\frac{1}{|x|}\right\}_{x=c} = (b-a)\left\{-\frac{1}{|c|^2}\right\}$$

   $$\Rightarrow \frac{1}{b} - 1 = (b+1)\left(-\frac{1}{c^2}\right) \Rightarrow c^2 = \frac{b^2+b}{b-1} \qquad (\because a = -1)$$

   $$\Rightarrow \frac{b^2+b}{b-1} < b^2 \quad (\because b^2 > c^2) \Rightarrow \frac{b(b+1)}{b-1} < b^2 \Rightarrow \frac{b+1}{b-1} < b$$

   $$\Rightarrow b + 1 < b^2 - b \quad \text{or} \quad b^2 - 2b - 1 > 0$$

   $$\Rightarrow (b-1)^2 > 2 \Rightarrow b - 1 > \sqrt{2} \Rightarrow b > 1 + \sqrt{2}.]$$

4. If in the Cauchy's mean value theorem, we take $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.

DARBOUX'S THEOREM ON DERIVATIVES

If a function f defined on [a, b] is such that

(*i*) *f is derivable on [a, b] and*

(*ii*) *f ′(a) and f ′(b) are of opposite signs*

then there exists at least one point c ∈ (a, b) such that f ′(c) = 0.

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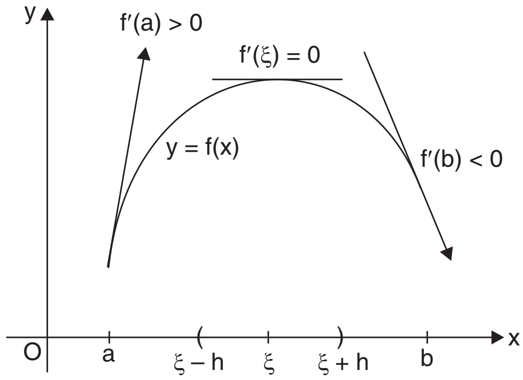
Proof. Since f is derivable on $[a, b]$, it is continuous on $[a, b]$ and hence f is bounded on $[a, b]$ and attains its bounds.



Let M = l.u.b. of f and m = g.l.b. of f on $[a, b]$.

Then, $\exists\ \xi, \eta \in [a, b]$ such that

$$M = f(\xi) \text{ and } m = f(\eta) \quad ...(1)$$

Since $f'(a)\, f'(b) < 0$, either

$f'(a) > 0$ and $f'(b) < 0$

or $\quad f'(a) < 0$ and $f'(b) > 0$.

Suppose $f'(a) > 0$ and $f'(b) < 0$.

Now, $f'(a) > 0 \Rightarrow$ there exists $\delta_1 > 0$ such that $f(x) > f(a)$ for $x \in (a, a + \delta_1)$ ...(2)

$f'(b) < 0 \Rightarrow$ there exists $\delta_2 > 0$ such that

$$f(x) > f(b) \text{ for } x \in (b - \delta_2, b) \quad ...(3)$$

$\therefore$ from (1), (2) and (3), we get

$$\xi \neq a \text{ and } \xi \neq b \text{ so that } a < \xi < b \quad ...(4)$$

Also, $\quad f(x) \leq f(\xi),\ \forall\ x \in [a, b]$. $\quad (\because\ f(\xi) = \text{l.u.b. of } f \text{ on } [a, b])$

Now, $a < \xi < b \Rightarrow$ there exists some $h > 0$ such that $a < \xi - h < \xi < \xi + h < b$.

$$\Rightarrow \quad f(\xi - h) - f(\xi) \leq 0 \quad \text{and} \quad f(\xi + h) - f(\xi) \leq 0$$

$$\Rightarrow \quad \frac{f(\xi - h) - f(\xi)}{-h} \geq 0 \quad \text{and} \quad \frac{f(\xi + h) - f(\xi)}{h} \leq 0, (h > 0)$$

Taking limits as $h \to 0$, we get

$$f'(\xi^-) \geq 0 \text{ and } f'(\xi^+) \leq 0$$

Since f is derivable on $[a, b]$, $f'(\xi)$ exists and $f'(\xi) = f'(\xi^-) = f'(\xi^+) = 0$.

i.e., $\quad f'(\xi) = 0,\ a < \xi < b$.

Hence, there exists $c\ (= \xi)$ such that $f'(c) = 0$, $c \in (a, b)$.

Similarly, we can prove that there exists $c(= \xi)$ such that $f'(c) = 0$, $c \in (a, b)$ when $f'(a) < 0$ and $f'(b) > 0$.

Remark. *Note the similarity between the above theorem and the intermediate value theorem.*

(Darboux's Intermediate Value Theorem for Derivatives)

Cor. 1. *If a function f is derivable on $[a, b]$ and $f'(a) \neq f'(b)$, and k is a number lying between $f'(a)$ and $f'(b)$; then there exists at least one point $c \in (a, b)$ such that $f'(c) = k$.*

Proof. Let k be any real number between $f'(a)$ and $f'(b)$.

Define a function F on $[a, b]$ such that $F(x) = f(x) - kx$.

Since f is derivable on $[a, b]$, F is also derivable on $[a, b]$ and $F'(x) = f'(x) - k$. Now, $F'(a) = f'(a) - k$ and $F'(b) = f'(b) - k$

Since k lies between $f'(a)$ and $f'(b)$, $F'(a)$ and $F'(b)$ are of opposite signs.

Hence, by Darboux's theorem, there exists $c \in (a, b)$ such that

$$F'(c) = 0 \quad \Rightarrow \quad f'(c) = k.$$

Cor. 2. *If f is a function derivable on a closed interval I, then image of I under f ′ is either an interval or a singleton.*

Let image of I under f' be J.

Since f is derivable on I, J $\neq \phi$.

Now, suppose J contains two points k_1 and k_2, then there exist $x_1, x_2 \in$ I such that

$$f'(x_1) = k_1 \text{ and } f'(x_2) = k_2.$$

If $k_1 \neq k_2$, then suppose k lies between k_1 and k_2. Then, by Darboux's theorem, there exists $c \in (x_1, x_2)$ such that $f'(c) = k$ and hence $k \in$ J.

Thus, $k_1, k_2 \in$ J, $\quad k_1 \neq k_2 \quad \Rightarrow \quad [k_1, k_2] \subseteq$ J.

$\Rightarrow$ J is an interval.

Cor. 3. *If $f'(x) \neq 0$ for all $x \in (a, b)$ then $f'(x)$ retains the same sign positive or negative on (a, b).*

Proof. Let, if possible, there exist $x_1, x_2 \in (a, b)$

such that $$f'(x_1) < 0 \text{ and } f'(x_2) > 0$$

$\therefore$ by Darboux's theorem, there exists $c \in (x_1, x_2) \subseteq (a, b)$ such that $f'(c) = 0$.

But this is against the given hypothesis that $f'(x) \neq 0$ for $x \in (a, b)$

Hence, $f'(x)$ retains the same sign for all $x \in (a, b)$.

Example 1. *If f is derivable on [a, b] then show that there is a point $c \in (a, b)$ such that*

$$f'(c) = \frac{1}{2}[f'(a) + f'(b)]$$

Sol. Let $$k = \frac{1}{2}[f'(a) + f'(b)]$$

Then, k lies between $f'(a)$ and $f'(b)$

$\therefore$ by Darboux's theorem, then there exist $c \in (a, b)$ such that $f'(c) = k$

$$\Rightarrow \quad f'(c) = \frac{1}{2}[f'(a) + f'(b)].$$

Example 2. *If a function f is derivable at a point c, show that | f | is also derivable at c, provided f(c) ≠ 0. What can you say if f(c) = 0 ? Justify.*

Sol. As the function f is derivable at the point c, f is continuous at point c.

Case (*i*). $f(c) \neq 0$:

If $f(c) > 0$, then there exists $\delta_1 > 0$ such that

$$f(x) > 0 \text{ for all } x \in (c - \delta_1, c + \delta_1)$$

If $f(c) < 0$, then there exists $\delta_2 > 0$ such that

$$f(x) < 0 \text{ for all } x \in (c - \delta_2, c + \delta_2)$$

Thus we have

$$f(x) > 0 \text{ for all } x \in (c - \delta_1, c + \delta_1)$$

and $$f(x) < 0 \text{ for all } x \in (c - \delta_2, c + \delta_2)$$

$$\Rightarrow \quad |f(x)| = \begin{cases} f(x) \text{ if } x \in (c - \delta_1, c + \delta_1 \\ -f(x) \text{ if } x \in (c - \delta_2, c + \delta_2) \end{cases}$$

Since f is derivable at the point c

$\therefore$ $| f |$ is also derivable at the point c

The result is not true if $f(c) = 0$. For example,

Consider $f(x) = x$ for all $x \in R$

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$\therefore$ $f(0) = 0$ and $| f(x) | = | x |$

Here f is derivable at $x = 0$ but $| f |$ is not derivable at $x = 0$.

HIGHER ORDER DERIVATIVES

We know that the existence of the derivative f' of a function f at a point c implies the existence and continuity of the function in a neighbourhood of c. The derivative of the function f' at c, if exists, is called the second derivative of f at c and is denoted by $f''(c)$. The existence of $f''(c)$ implies the existence and continuity of f' in a neighbourhood of c. Similar is the case for higher order derivatives. The nth derivative of f at c is denoted by $f^n(c)$.

Taylor's Theorem*. *If a function f defined on [a, a +h] is such that*

(*i*) *the (n – 1)th derivative f^{n-1} is continuous on [a, a + h] and*

(*ii*) *the nth derivative f^n exists on (a, a + h), then there exists a real number θ, 0 < θ < 1, such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{p((n-1)!)} f^n(a+\theta h) \quad \ldots(1)$$

where p is a given positive integer.

Proof. Since f^{n-1} exists, all the derivatives $f', f'', \ldots, f^{n-1}$ exist and continuous on $[a, a + h]$. Consider a function ϕ defined on $[a, a + h]$ such that

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x)^p \quad \ldots(2)$$

where A is a constant to be determined such that

$$\phi(a+h) = \phi(a)$$

But, $\phi(a+h) = f(a+h)$

and $$\phi(a) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah^p$$

$$\therefore \quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah^p \quad \ldots(3)$$

(*i*) Now, $f, f', f'', \ldots, f^{n-1}$ being all continuous on $[a, a + h]$, the function ϕ is continuous on $[a, a + h]$;

*After the name of an English Mathematician Brook Taylor (1685—1731).

(*ii*) the functions $f, f', ..., f^{n-1}$ and $(a + h - x)^r$ for all r, being all derivable on $(a, a + h)$, the function ϕ is derivable on $(a, a + h)$; and

(*iii*) $\phi\,(a + h) = \phi(a)$.

Thus, the function ϕ satisfies all the conditions of Rolle's theorem and hence $\exists$ at least one real number θ, $0 < \theta < 1$, such that

$$\phi'(a + \theta h) = 0.$$

But, from (2),

$$\phi'(x) = f'(x) + [-f'(x) + (a + h - x)\, f''(x)] + \frac{1}{2!}[-2\,(a + h - x)\, f''(x) + (a + h - x)^2 f'''(x)] + ...$$
$$+ \frac{1}{(n-1)!}[-(n-1)(a + h - x)^{n-2} f^{n-1}(x) + (a + h - x)^{n-1} f^n(x)] - Ap(a + h - x)^{p-1}$$
$$= \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - Ap(a + h - x)^{p-1}$$

(Other terms cancel in pairs)

$$\therefore \qquad 0 = \phi(a + \theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a + \theta h) - Aph^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \qquad A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a + \theta h),\ h \neq 0,\ \theta \neq 1$$

Substituting A from (3) in (2), we get the required result.

Another form of Taylor's Theorem

If a function f defined on [a, b] is such that

(*i*) *the (n – 1)th derivative of f^{n-1} is continuous on [a, b] and*

(*ii*) *the nth derivative of f^n exists on (a, b)*

then there exists a number c, a < c < b, such that

$$f(b) = f(a) + (b - a)\, f'(a) + \frac{1}{2!}(b - a)^2 f''(a) + ...$$
$$+ \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!}(b-a)^n f^{(n)}(c).$$

[For proof, apply Rolle's theorem to the function

$$\phi(x) = f(b) - f(x) - (b - x)\, f'(x) - \frac{1}{2!}(b - x)^2 f''(x) - ......$$
$$- \frac{1}{(n-1)!}(b-x)^{n-1} f^{(n-1)}(x) - \frac{1}{n!}(b-x)^n A$$

where A is a constant such that $\phi(a) = \phi(b)$]

Forms of Remainder after n terms :

(*i*) the term $R_n = \dfrac{h^n(1-\theta)^{n-p}}{p((n-1)!)} f^n(a + \theta h)$,

which occurs after n terms, is called the Taylor's remainder after n terms. The theorem with this form of remainder is called Taylor's theorem with *Schlomilch* and *Roche form of remainder*.

(*ii*) For $p = 1$, we get $\quad R_n = \dfrac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n (a + \theta h)$

and is called *Cauchy's form of remainder*.

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(*iii*) For $p = n$, we get $\quad R_n = \dfrac{h^n}{n!} f^n (a + \theta h)$

and is called *Lagrange's form of remainder*.

Second form of Taylor's Theorem. Replacing $a + h$ by x (or h by $x - a$) in (1), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + ... + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$
$$+ \frac{(x-a)^n}{p((n-1)!)} (1-\theta)^{n-p} f^n(a + \theta (x-a))$$
$$\text{where } 0 < \theta < 1 \qquad ...(4)$$

The remainder after n terms can be written as

$$R_n = \frac{(x-a)^n (1-\theta)^{n-p}}{p((n-1)!)} f^n(c), \; a < c < x.$$

Maclaurin's Theorem. Putting $a = 0$ in (4), we have for $x \in (0, h)$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-p}}{p((n-1)!)} f^n(\theta x)$$

and is called *Maclaurin's theorem with Schlomilch and Roche form of remainder*.

For $p = 1$, $R_n = \dfrac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ and is *Cauchy's form of remainder*.

For $p = n$, $R_n = \dfrac{x^n}{n!} f^n(\theta x)$ and is *Lagrange's form of remainder*.

Some More Results Concerning Taylor's Theorem

(*i*) *If a function f defined on [a, b] satisfies all the conditions of Taylor's theorem then*

$$f(a) = f(b) + (a-b) f'(b) + \frac{1}{2!} (a-b)^2 f''(b) + ...$$
$$+ \frac{1}{(n-1)!} (a-b)^{n-1} f^{(n-1)}(b) + \frac{1}{n!} (a-b)^n f^{(n)}(c), \; a < c < b.$$

[**Hint.** Apply Lagrange's mean value theorem to the function

$$F(x) = f(a) - f(x) - (a-x) f'(x) - \frac{1}{2!} (a-x)^2 f''(x) - ......$$
$$- \frac{1}{(n-1)!} (a-x)^{n-1} f^{(n-1)}(x) - \frac{1}{n!} (a-x)^n A$$

where A is a constant such that $F(a) = F(b)$.]

(*ii*) *If a function f satisfies all the conditions of Taylor's theorem on [a – h, a], h > 0, then*

$$f(a-h) = f(a) - hf'(a) + \frac{1}{2!} h^2 f''(a) + ...$$
$$+ \frac{(-1)^{n-1}}{(n-1)!} h^{n-1} f^{(n-1)}(a) + \frac{(-1)^n}{n!} h^n f^{(n)}(a - \theta h), \; 0 < \theta < 1$$

[Proof. Let $c = a - h$, so that $c < a$ and f satisfies the conditions of Taylor's theorem on $[c, a]$. Therefore, expressing $f(c)$ in terms of values at the upper end, of f and its derivatives,

$$f(c) = f(a) + (c - a) f'(a) + \frac{1}{2!} (c - a)^2 f''(a) + \ldots..$$
$$+ \frac{1}{(n-1)!} (c - a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!} (c - a)^n f^{(n)}(d),\ c < d < a. \quad \ldots(1)$$

Since $c = a - h$, $h > 0$, therefore $c - a = -h$.

Also, $c < d < a \iff a - h < d < a \iff -h < d - a < 0$

$$\iff \quad 0 < \frac{d-a}{-h} < 1 \quad \ldots(2)$$

Put $\frac{d-a}{-h} = \theta$ so that $0 < \theta < 1$ and $d = a - \theta h$.

$\therefore$ Using values of c and d in (1), we get

$$f(a - h) = f(a) - hf'(a) + \frac{1}{2!} h^2 f''(a) + \ldots\ldots$$
$$+ \frac{(-1)^{n-1}}{(n-1)!} h^{n-1} f^{(n-1)}(a) + \frac{(-1)^n}{n!} f^{(n)}(a - \theta h),\ 0 < \theta < 1]$$

(*iii*) *If a function f satisfies the conditions of Taylor's theorem on $[a - h, a]$, $h > 0$ and $x \in [a - h, a]$ then*

$$f(x) = f(a) + (x - a) f'(a) + \frac{1}{2!} (x - a)^2 f''(a) + \ldots\ldots$$
$$+ \frac{1}{(n-1)!} (x - a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!} (x - a)^n f^{(n)}(a + \theta(x - a)),\ 0 < \theta < 1.]$$

[Proof. Let $x \in [a - h, a]$ so that $[x, a] \subseteq [a - h, a]$ and f satisfies the conditions of Taylor's theorem on $[x, a]$.

Now, keeping x fixed, and using Taylor's theorem to express the value of f at the lower end x of $[x, a]$, in terms of the values of f and its derivatives at the upper end a, we have

$$f(x) = f(a) + (x - a) f'(a) + \frac{1}{2!} (x - a)^2 f''(a) + \ldots\ldots$$
$$+ \frac{1}{(n-1)!} (x - a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!} (x - a)^n f^{(n)}(a + \theta(x - a)),\quad 0 < \theta < 1.]$$

(*iv*) *If a function f satisfies all the conditions of Taylor's theorem on $[a - h, a + h]$ and $x \in [a - h, a + h]$, then*

$$f(x) = f(a) + (x - a) f'(a) + \frac{1}{2!} (x - a)^2 f''(a) + \ldots\ldots$$
$$+ \frac{1}{(n-1)!} (x - a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!} (x - a)^n f^{(n)}[a + \theta(x - a)],\ 0 < \theta < 1.$$

[Proof. Since f satisfies all the conditions of Taylor's theorem on $[a - h, a + h]$, $h > 0$ and hence on $[a - h, a]$ and $[a, a + h]$ also.

Now, $x \in [a - h, a + h]$ iff $x \in [a - h, a]$ or $x \in [a, a + h]$

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In either case, we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + ... + \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a)$$

$$+ \frac{1}{n!}(x-a)^n f^{(n)}[a + \theta(x-a)],\ 0 < \theta < 1]$$

(*v*) **Maclaurin's theorem.** *If a function f defined on [– h, h], h > 0 such that*

(*i*) *$f^{(n-1)}$ is continuous on [– h, h] and*

(*ii*) *$f^{(n)}$ is derivable on (– h, h), then for $x \in [-h, h]$, there exist θ, 0 < θ < 1 such that*

$$f(x) = f(0) + x f'(0) + \frac{1}{2!} x^2 f''(0) + ...$$

$$+ \frac{1}{(n-1)!} x^{n-1} f^{(n-1)}(0) + \frac{1}{n!} x^n f^{(n)}(\theta x).$$

[**Proof.** Put $a = \theta$ in the above result.]

Taylor's and Maclaurin's infinite series

Let a function *f* possess continuous derivatives of every order in $[a, a + h]$, then for all $n \in N$, we have by Taylor's theorem

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ...$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h),\ (0 < \theta < 1).$$

$$= S_n + R_n$$

where $\quad S_n = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$

and $\quad R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h).$

Now, if $R_n \to 0$ as $n \to \infty$, then $S_n \to f(a + h)$

$\Rightarrow$ the infinite series $f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + ...$ converges to $f(a + h)$.

Thus, if

(*i*) *f* possesses continuous derivatives of every order in $[a, a + h]$

and (*ii*) $R_n \to 0$ as $n \to \infty$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^{(n)}(a) + ...$$

The infinite series on R.H.S. is called Taylor's series.

If we put $a = 0$ and replace *h* by *x*, then we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^{(n)}(0) + ...$$

The infinite series on R.H.S. is called Maclaurin's series.

Here R_n can be of any form.

Example 1. *If f is derivable for each x ∈ R and* $|f(x)| < A$, $|f''(x)| < B$, *A, B being constants, prove that* $|f'(x)| < 2\sqrt{AB}$.

Sol. Applying Taylor's theorem with Lagrange's form of remainder after 2 terms to f in $[x, x+h]$, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x+\theta h),\ 0 < \theta < 1.$$

$$\Rightarrow \qquad |h f'(x)| = \left| f(x+h) - f(x) - \frac{h^2}{2!} f''(x+\theta h) \right|$$

$$\leq |f(x+h)| + |f(x)| + \frac{1}{2} h^2 |f''(x+\theta h)|$$

$$\Rightarrow \qquad |f'(x)| \leq \frac{1}{h}\left\{A + A + \frac{1}{2}h^2 B\right\} = \frac{2A}{h} + \frac{h}{2} B$$

∴ $|f'(x)|$ is independent of h and less than

$$\frac{2A}{h} + \frac{h}{2} B \text{ for all } h > 0 \qquad \text{...(1)}$$

∴ $|f'(x)|$ is less than the least value of $\frac{2A}{h} + \frac{hB}{2}$.

Let $$g(h) = \frac{2A}{h} + \frac{hB}{2} = \left(\sqrt{\frac{2A}{h}} - \sqrt{\frac{Bh}{2}}\right)^2 + 2\sqrt{AB}$$

∴ Minimum value of $g(h)$ is $2\sqrt{AB}$.

∴ From (1), $|f'(x)| < 2\sqrt{AB}$.

Example 2. *Show that the number* θ, *which occurs in Taylor's formula for f at a with Lagrange's form of remainder after n terms, approaches the limit* $\frac{1}{n+1}$ *as* $x \to a$ *provided* $f^{(n+1)}$ *is continuous at a and* $f^{(n+1)}(a) \neq 0$.

Sol. Since $f^{(n+1)}$ is continuous at a, there exists $\delta > 0$ such that $f^{(n+1)}(x)$ exists on $(a-\delta, a+\delta)$. With $h < \delta$, $f^{(n+1)}(x)$ exists on $[a-h, a+h]$ and hence on $[a, a+h]$ also. Therefore, $f^{(n)}(x)$ and all lower order derivatives are continuous and derivable on $[a, a+h]$.

∴ by Taylor's formula for f at a with Lagrange's form of remainder after n terms, we have

$$f(a+h) = f(a) + h f'(a) + \frac{1}{2!} h^2 f''(a) + \ldots\ldots$$

$$+ \frac{1}{(n-1)!} h^{n-1} f^{(n-1)}(a) + \frac{1}{n!} h^n f^{(n)}(a+\theta h),\ 0 < \theta < 1. \qquad \text{...(1)}$$

This defines θ in terms of h.

Again, applying Taylor's formula for f at a, with Lagrange's form of remainder after $n+1$ terms, we have

$$f(a+h) = f(a) + h f'(a) + \frac{1}{2!} h^2 f''(a) + \ldots\ldots$$

$$+ \frac{1}{n!} h^n f^{(n)}(a) + \frac{1}{(n+1)!} h^{n+1} f^{(n+1)}(a+\theta_1 h),\ 0 < \theta < 1 \qquad \text{...(2)}$$

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$\therefore$ from (1) and (2), we get

$$\frac{1}{n!} h^n f^{(n)} (a + \theta h) = \frac{h^n}{n!} f^{(n)} (a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)} (a + \theta_1 h).$$

$$\Rightarrow \quad f^{(n)} (a + \theta h) - f^{(n)} (a) = \frac{h}{n+1} f^{(n+1)} (a + \theta_1 h) \qquad (\because \ h > 0) \quad ...(3)$$

Since $f^{(n)}$ is continuous and differentiable on $[a, a + \theta h]$, $0 < \theta < 1$, we have by Lagrange's Mean Value Theorem,

$$\theta h f^{(n+1)} (a + \theta \theta_2 h) = \frac{h}{n+1} f^{(n+1)} (a + \theta_1 h)$$

Cancelling h, we have

$$\theta = \frac{1}{n+1} \frac{f^{(n+1)} (a + \theta_1 h)}{f^{(n+1)} (a + \theta \theta_2 h)}$$

$$\Rightarrow \quad \lim_{h \to 0} \theta = \frac{1}{n+1}$$, provided $f^{(n+1)} (x)$ is continuous at a and $f^{(n+1)} (a) \neq 0$.

Example 3. *Using Taylor's theorem, prove that*

$$x - \frac{x^2}{2} < \log (1 + x) < x - \frac{x^2}{2(1+x)^2}, \ x > 0.$$

Sol. Let $\quad f(x) = \log (1 + x), x \geq 0.$

Let $a > 0$ so that $[0, a] \subseteq [0, \infty)$.

The function f is continuous and derivable on $[0, a]$ and $f'(x) = \frac{1}{1+x}$, $x \in [0, a]$ and $f'(0) = 1$.

The function f' is again derivable on $[0, a]$ and $f''(x) = -\frac{1}{(1+x)^2}$

$\therefore$ by Taylor's theorem for f on $[0, x] \subseteq [0, a]$, with Lagrange's form of remainder after 2 terms, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f'' (\theta x)$$

$$\Rightarrow \quad \log (1 + x) = 0 + x - \frac{x^2}{2} \cdot \frac{1}{(1+\theta x)^2} = x - \frac{x^2}{2(1+\theta x)^2}, \ 0 < \theta < 1.$$

$$\Rightarrow \quad \frac{1}{(1+\theta x)^2} = \frac{-2}{x^2} [\log (1 + x) - x]$$

Since 'a' is arbitrary, it is true for all $x > 0$.

Also, $0 < \theta < 1$ and $x > 0 \ \Rightarrow \ 1 < 1 + \theta x < 1 + x$

$$\therefore \quad \frac{1}{1+x} < \frac{1}{1+\theta x} < 1 \text{ and hence } \frac{1}{(1+x)^2} < \frac{1}{(1+\theta x)^2} < 1$$

$$\Rightarrow \quad \frac{1}{(1+x)^2} < \frac{-2}{x^2} [\log (1 + x) - x] < 1.$$

$$\Rightarrow \quad \frac{-x^2}{2(1+x)^2} > \log(1+x) - x > -\frac{x^2}{2}$$

$$\Rightarrow \quad x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)^2}, \; x > 0.$$

Example 4. *If f is continuous on [a, b] and possesses finite first and second order derivatives for* $x = x_0$*, where* $a < x_0 < b$*, prove that*

$$f''(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}.$$

Sol. Since $f'(x_0)$ and $f''(x_0)$ exist, they also exist in the neighbourhood $(x_0 - h, x_0 + h)$, $h > 0$. Hence, applying Taylor's theorem for the intervals $[x_0, x_0 + h]$ and $[x_0 - h, x_0]$, we get

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0 + \theta_1 h), \; 0 < \theta_1 < 1$$

and $$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!} f''(x_0 - \theta_2 h), \; 0 < \theta_2 < 1$$

Adding, we get

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + \frac{h^2}{2!} [f''(x_0 + \theta_1 h) + f''(x_0 - \theta_2 h)]$$

$$\Rightarrow \quad \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = \frac{1}{2} [f''(x_0 + \theta_1 h) + f''(x_0 - \theta_2 h)]$$

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

Example 5. *Show that if*

$$-1 < x < 1, \; (1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{23}x^4 + R_5$$

where $R_5 = \dfrac{x^5}{4!}(1-\theta)^4\left(-\dfrac{880}{243}\right)(1+\theta x)^{-14/3}.$

Deduce the value of $9^{1/3}$ *to four decimal places and estimate the error involved.*

Sol. $$f(x) = (1+x)^{1/3}, \; -1 < x < 1 \qquad \ldots(1)$$

Taylor's expansion of f at $x = 0$ with Cauchy's form of remainder is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$$
$$+ \frac{x^5}{5!}(1-\theta)^4 f^{v}(\theta x), \; 0 < \theta < 1 \qquad \ldots(2)$$

Now, $f(x) = (1+x)^{1/3}$ $\qquad f(0) = 1$

$f'(x) = \dfrac{1}{3}(1+x)^{-2/3}$ $\qquad f'(0) = \dfrac{1}{3}$

$f''(x) = \dfrac{1}{3}\left(\dfrac{-2}{3}\right)(1+x)^{-5/3}$ $\qquad f''(0) = \dfrac{-2}{9}$

$f'''(x) = \dfrac{1}{3}\left(\dfrac{-2}{3}\right)\left(\dfrac{-5}{3}\right)(1+x)^{-8/3}$ $\qquad f'''(0) = \dfrac{10}{27}$

$$f^{iv}(x) = \frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{-8}{3}\right)(1+x)^{\frac{-11}{3}} \qquad f^{iv}(x) = \frac{-80}{81}$$

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$$f^{v}(x) = \frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{-8}{3}\right)\left(\frac{-11}{3}\right)(1+x)^{-\frac{14}{3}}$$

$$= \frac{880}{243}(1+x)^{\frac{-14}{3}} \qquad \ldots(3)$$

Therefore, from (2), we get

$$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{9}\frac{x^2}{2!} + \frac{10}{27}\frac{x^3}{3!} - \frac{80}{81}\frac{x^4}{4!} + R_5$$

Keeping first five terms only, we get

$$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 \qquad \ldots(4)$$

Now, $\quad 9^{1/3} = (8+1)^{1/3} = 2(1+.125)^{1/3}$

$\therefore$ Putting $x = .125$ in (4), we get

$$9^{1/3} = 2\left[1 + \frac{1}{3}(.125) - \frac{1}{9}(.125)^2 + \frac{5}{81}(.125)^3 - \frac{10}{243}(.125)^4\right]$$

$$= 2\left[1 + .04166 - \frac{1}{9}(.01562) + \frac{5}{81}(.00195) - \frac{10}{243}(.00024)\right]$$

$$\simeq 2.08008$$

Error involved $= |R_5|$

$$= \left|\frac{x^5(1-\theta)^4}{4!}\left(\frac{880}{243}\right)\left(1+\frac{\theta}{8}\right)^{-14/3}\right|$$

$$\leq (.125)^5 \frac{1}{4!} \times \left(\frac{880}{243}\right) \times 1 \qquad (\because \quad x = .125)$$

$$= (.125)^5 \times \frac{110}{729} < .00001.$$

Note. *In the above example, we had $x = .125 > 0$ and there was no difficulty in estimating the error. We could use Lagrange's form of remainder also. But, since R_n contains negative powers of $(1 + \theta x)$ and therefore it will create some difficulty when $x < 0$.*

In such cases, Cauchy's form of remainder is really useful.

Example 6. *Using Maclaurin's theorem, find the expansion of $\cos x$, $x \in R$ about $x = 0$.*

Sol. Let $\quad f(x) = \cos x, \ x \in \mathrm{R}$

$f(x)$ is continuous as well as derivable on R.

and $$f'(x) = -\sin x = \cos\left(x + \frac{\pi}{2}\right), \ x \in \mathrm{R}, \ f'(0) = 0.$$

Clearly, $\quad f^{(m)}(x) = \cos\left(x + \dfrac{m\pi}{2}\right), \ x \in \mathrm{R}, \ m = 1, 2, 3, \ldots\ldots$

$\Rightarrow \quad f^{(m)}(0) = \cos\dfrac{m\pi}{2}$.

Now, for any real number A > 0, [– A, A] ⊆ R. Thus, for any positive integer n, $f^{(n-1)}(x)$ is continuous and derivable for all $x \in [-A, A]$,

$$f(x) = f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \ldots\ldots$$

$$+ \frac{1}{(n-1)!}x^{n-1} f^{(n-1)}(0) + \frac{1}{n!}x^n f^{(n)}(\theta x),\ \ 0 < \theta < 1$$

$$= S_n(x) + R_n(x),$$

where $R_n(x) = \dfrac{1}{n!}x^n f^{(n)}(\theta x) = \dfrac{x^n}{n!}\cos\left(\theta x + \dfrac{n\pi}{2}\right)$

$$\Rightarrow \quad |R_n(x)| = \frac{|x|^n}{n!}\left|\cos\left(\theta x + \frac{n\pi}{2}\right)\right| \le \frac{A^n}{n!} \qquad (\because\ |x| < A)$$

Let $\quad a_n = \dfrac{A^n}{n!} \Rightarrow \dfrac{a_{n+1}}{a_n} = \dfrac{A}{n+1} \Rightarrow \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = 0 < 1$

∴ the sequence $< a_n >$ is convergent and $a_n \to 0$ as $n \to \infty$

$$\therefore \quad \lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty}\frac{A^n}{n!} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} R_n(x) = 0,\ x \in [-A, A]$$

Since A > 0 is arbitrary, the expansion of cos x is valid for all x ∈ R, and

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{1}{(n-1)!}x^{n-1} f^{(n-1)}(0) + \ldots\ldots$$

$$= 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!}\cos(n-1)\frac{\pi}{2} + \ldots\ldots$$

When $n = 2k + 1$, then $\cos(n-1)\dfrac{\pi}{2} = \cos k\pi = (-1)^k$.

$$\therefore \quad f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots + (-1)^k\frac{x^{2k}}{(2k)!} + \ldots\ldots,\ x \in R.$$

Example 7. *Find the expansion of log (1 + x), – 1 < x < 1 about x = 0.*

Sol. Let $\quad f(x) = \log(1 + x),\ x > -1$

Clearly, $f(x)$ is continuous and derivable for all $x > -1$ and $f'(x) = \dfrac{1}{1+x}$, $x > -1$ ...(1)

Also, for each positive integer n, $f^{(n)}(x)$ exists.

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n},\ x > -1$$

and $\quad f^{(n)}(0) = (-1)^{n-1}.(n-1)!$ ...(2)

Clearly, $f^{(n)}(x)$ is continuous and derivable for all $x > -1$.

Let $a, b \in R$ such that $-1 < a < 0$ and $b > 0$ then, $[a, b] \subseteq [-1, \infty)$

Now, $f^{(n-1)}(x)$ is continuous and derivable on $[a, b]$.

Also, $[0, b] \subseteq [a, b]$.

$\therefore$ by Maclaurin's theorem for f on $[0, b]$ with Lagrange's form of remainder after n terms and $x \in [0, b]$.

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$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots$$

$$+ \frac{1}{(n-1)!} x^{n-1} f^{(n-1)}(0) + \frac{1}{n!} x^n f^{(n)}(\theta x), \quad 0 < \theta < 1$$

$$= S_n(x) + R_n(x) \qquad \ldots(3)$$

where $\quad |R_n(x)| = \left| \frac{x^n}{n!} f^{(n)}(\theta x) \right| = \frac{|x|^n}{n!} \left| \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} \right|$

$$\leq \frac{b^n}{n!}(n-1)! = \frac{b^n}{n!} \qquad \left(\because \; x > 0, \theta > 0, \frac{1}{1+\theta x} < 1 \right)$$

Let $\qquad a_n = \frac{b^n}{n} \quad \Rightarrow \quad \frac{a_{n+1}}{a_n} = \frac{nb}{n+1} \to b$ as $n \to \infty$

$\therefore \quad < a_n >$ is convergent if $b < 1$.

$\therefore$ When $b < 1$, $\displaystyle \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{b^n}{n} = 0 \;\Rightarrow\; \lim_{n\to\infty} R_n(x) = 0.$

Hence, expansion of log $(1 + x)$ is valid for all x, $0 \leq x < 1$. ...(4)

Again, since $[a, 0] \subseteq [a, b]$.

$\therefore$ by Maclaurin's theorem for f on $[a, 0]$, with Cauchy's form of remainder after n terms and with $x \in [a, 0]$

$$f(x) = f(0) + x f'(0) + \frac{1}{2!} x^2 f''(0) + \ldots\ldots.$$

$$+ \frac{1}{(n-1)!} x^{n-1} f^{(n-1)}(0) + \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x), \; 0 < \theta < 1$$

$$= S_n(x) + R_n(x) \qquad \ldots(5)$$

where $\quad |R_n(x)| = \left| \frac{x^n (1-\theta)^{n-1} f^{(n)}(\theta x)}{(n-1)!} \right| = \frac{|x|^n (1-\theta)^{n-1} |(-1)^{n-1}|}{|1+\theta x|^n} \qquad \ldots(6)$

Now, $a < x < 0 \;\Rightarrow\; |1 + \theta x| \geq |1 - |\theta x|| \geq |1 - \theta|x|| > |1 - \theta|$

$(\because \; |x| \leq |a| < 1)$

$\therefore \qquad |R_n(x)| \leq \frac{|a|^n (1-\theta)^{n-1}}{(1-\theta)^n} = \frac{|a|^n}{(1-\theta)} \to 0$ as $n \to \infty \qquad (\because \; |a| < 1)$

$\Rightarrow \qquad \displaystyle \lim_{n\to\infty} R_n(x) = 0.$

$\therefore$ expansion of log $(1 + x)$ is valid for all x, $-1 < x < 0$ ...(7)

Hence from (4) and (7), we see that the expansion of log $(1 + x)$ about $x = 0$ is valid for all $x \in (-1, 1)$ and we have

$$f(x) = f(0) + x f'(0) + \frac{1}{2!} x^2 f''(0) + \ldots\ldots + \frac{1}{(n-1)!} x^{n-1} f^{(n-1)}(0) + \ldots\ldots$$

$$= 0 + x \,.\, 1 + \frac{1}{2\,!}\, x^2 \,.\, (-1) \,.\, 1\,! + \frac{1}{3\,!}\, x^3 \,.\, (-1)^2 \,.\, 2\,!$$

$$+ \ldots\ldots + \frac{x^{n-1}}{(n-1)\,!}\,(-1)^{n-2} \,.\, (n-2)\,! + \ldots\ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + \frac{(-1)^{n-2}}{(n-1)}\, x^{n-1} + \ldots\ldots, \; -1 < x < 1.$$

EXERCISE 3

1. Prove Taylor's theorem with Lagrange's form of remainder by considering the function $\phi(x) = f(x) + (a + h - x)\, f'(x) + \frac{(a+h-x)^2}{2\,!}\, f''(x) + \ldots. + \frac{(a+h-x)^{n-1}}{(n-1)\,!}\, f^{n-1}(x) + A(a + h - x)^n$ on $[a, a + h]$.
2. Prove Taylor's theorem with Cauchy's form of remainder by considering the function $\phi(x) = f(x) + (a + h - x)\, f'(x) + \frac{(a+h-x)^2}{2\,!}\, f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)\,!}\, f^{n-1}(x) + A(a + h - x)$ on $[a, a + h]$.
3. Applying Lagrange's Mean Value Theorem to the function $\phi(x) = f(x) + (a + h - x)\, f'(x) + \frac{(a+h-x)^2}{2\,!}\, f''(x) + \ldots\ldots. + \frac{(a+h-x)^{n-1}}{(n-1)\,!}\, f^{n-1}(x) + Ah^p$ on $[a, a + h]$ to prove Taylor's theorem with Cauchy form of remainder.

   Remark. *For n = 1, the theorem reduces to the Mean Value Theorem. For this reason, Taylor's theorem is also called the Generalized Mean Value Theorem.*
4. State and prove Taylor's theorem.
5. If $f(x + h) = f(x) + hf'(x) + \frac{1}{2}\, h^2 f''(x + \theta h)$ and $f(x) = x^3$, show that $\theta = \frac{1}{3}$.
6. Show that if $a > 0$, $h > 0$ and $n \in$ N, then there exists θ, $0 < \theta < 1$ such that

$$\frac{1}{a+h} = \frac{1}{a} - \frac{h}{a^2} + \frac{h^2}{a^3} - \ldots\ldots + \frac{(-1)^{n-1}\, h^{n-1}}{a^n} + R_n, \qquad \text{where } R_n = \frac{(-1)^n\, h^n}{(a+\theta h)^{n+1}}\,.$$

   [**Hint.** Apply Taylor's theorem with Lagrange's form of remainder to $f(x) = \frac{1}{x}$, on $[a, a + h]$ at $x = a$.]
7. Find

   (*i*) expansion of e^x (*ii*) expansion of sin x.
8. If a function f is such that its derivative f' is continuous on $[a, b]$ and differentiable on (a, b) then show that there exists a number c between a and b such that

$$f(b) = f(a) + (b - a)\, f'(a) + \frac{1}{2}\,(b - a)^2 f''(c)$$

   [**Hint.** Consider the function

$$\phi(x) = f(x) + (b - x)\, f'(x) + (b - x)^2 A$$

   where A is a constant to be determined such that

   $\phi(a) = \phi(b)$. Apply Rolle's theorem to $\phi(x)$]
9. If a function is twice differentiable on $[a, a + h]$, then show that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2}\, f''(a + \theta h),\; 0 < \theta < 1$$

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[**Hint.** Consider the function

$$\phi(x) = f(x) + (a + h - x)\, f'(x) + (a + h - x)^2\, A.$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$. Apply Rolle's theorem to $\phi(x)$].

Answers

7. (*i*) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} + \ldots\ldots,\ x \in \mathbb{R}$$

(*ii*) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots + \frac{(-1)^{k-1}}{(2k-1)!}\, x^{2k-1} + \ldots\ldots,\ x \in \mathbb{R}.$$

UNIT II

NOTES

2. GENERAL THEOREMS

STRUCTURE

Introduction
Taylor's Theorem with Lagrange's Form of Remainder After n Terms
Taylor's Theorem with Cauchy's Form of Remainder

INTRODUCTION

In the present chapter, we shall consider a few general theorems which play a very important role in the subsequent development of the subject matter of Differential Calculus.

TAYLOR'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER AFTER n TERMS

If a function $f(x)$ be such that

(*i*) $f(x), f'(x), f''(x) \ldots\ldots f^{n-1}(x)$ are continuous in the closed interval $[a, a+h]$ and

(*ii*) $f^n(x)$ exists in the open interval $(a, a+h)$, then there exists at least one number θ between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

Proof. Consider the function

$$\mathrm{F}(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots$$

$$\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(\mathbf{a}+\mathbf{h}-\mathbf{x})^{\mathbf{n}}}{\mathbf{n}!} . \mathbf{A} \quad \ldots(1)$$

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where A is a constant to be so chosen that

$$F(a + h) = F(a).$$

Put $x = a + h$ in (1),

$\therefore$ $\quad F(a + h) = f(a + h)$ (All other terms becomes zero)

Put $x = a$ in (1),

$$F(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A$$

Putting these values of $F(a + h)$ and $F(a)$ in $F(a + h) = F(a)$, we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \quad ...(2)$$

which gives the value of A.

Now let us apply Rolle's Theorem on F(x) in [a, a + h]

It is given that $f(x), f'(x), f''(x), ... f^{n-1}(x)$ are continuous in the closed interval $[a, a + h]$ and their derivatives $f'(x), f''(x), ... f^n(x)$ all exist in the open interval $(a, a + h)$.

Also $(a + h - x), \frac{(a+h-x)^2}{2!}, ... \frac{(a+h-x)^n}{n!}$ being polynomials and A are all continuous in the closed interval $[a, a + h]$ and derivable in open interval $(a, a + h)$. Therefore, $F(x)$ is continuous in the closed interval $[a, a + h]$ and derivable in open interval $(a, a + h)$.

Also $\quad F(a + h) = F(a)$.

Thus, $F(x)$ satisfies all the three conditions of Rolle's theorem, and therefore, there must exist at least one positive number θ less than 1 such that

$$F'(a + \theta h) = 0 \; [0 < \theta < 1] \quad ...(3)$$

Differentiating both sides of (1) w.r.t. x ;

$$F'(x) = f'(x) + [(a + h - x) f''(x) - f'(x)] + \frac{1}{2!} [(a + h - x)^2 f'''(x) - 2(a + h - x) f''(x)]$$

$$...... + \frac{1}{(n-1)!} [(a + h - x)^{n-1} f^n(x) - (n-1)(a + h - x)^{n-2} f^{n-1}(x)]$$

$$- \frac{nA}{n!} (a + h - x)^{n-1}$$

or $\quad F'(x) = \frac{1}{(n-1)!} (a + h - x)^{n-1} f^n(x) - \frac{A}{(n-1)!} (a + h - x)^{n-1}$

[$\because$ other terms cancel in pairs]

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^n(x) - A]$$

Putting $x = a + \theta h$,

$$F'(a + \theta h) = \frac{(h - \theta h)^{n-1}}{(n-1)!} [f^n(a + \theta h) - A] = 0 \quad \text{[By (3)]}$$

i.e., $\quad \frac{h^{n-1}}{(n-1)!} (1 - \theta)^{n-1} [f^n(a + \theta h) - A] = 0$

$\therefore$ $\quad f^n(a + \theta h) = A \quad$ [$\because$ neither $h = 0$, nor $(1 - \theta) = 0$ as $0 < \theta < 1$]

NOTES

Putting this value of A in (2), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h) \quad [0 < \theta < 1]$$

Note. The $(n + 1)$th term $\frac{h^n}{n!} f^n(a + \theta h)$ in the above expansion is in general, denoted by R_n and is called *Lagrange's remainder after n terms.*

Cor. If we put $n = 1$ in the above theorem, it takes the form

$$f(a + h) = f(a) + hf'(a + \theta h) \text{ where } 0 < \theta < 1$$

which is Lagrange's mean value theorem and hence this shows *that mean-value theorem is a particular case of Taylor's theorem.*

Alternative form of Taylor's theorem with Lagrange's form of remainder after n terms

If we put $a + h = b$, then interval $[a, a + h]$ becomes $[a, b]$ and $a + \theta h = a + (b - a)\theta = c$, where c lies between a and b, and theorem becomes

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c) \quad [a < c < b]$$

Cor. Maclaurin's theorem with Lagrange's form of remainder after n terms.

Proof. Firstly, state and prove Taylor's Theorem with Lagrange's form of Remainder as in Art. 2. Then put $a = 0$ and $h = x$.

Hence we get

$$\mathbf{f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) \ldots + \frac{x^n}{n!} f^n(\theta x) \quad [0 < \theta < 1]}$$

which is called **Maclaurin's theorem with Lagrange's form of remainder** and holds good when

(*i*) $f(x)$, $f'(x)$, $f''(x)$ ...... $f^{n-1}(x)$ are continuous in the closed interval $[0, x]$

and (*ii*) $f^n(x)$ exists in the open interval $(0, x)$.

TAYLOR'S THEOREM WITH CAUCHY'S FORM OF REMAINDER

If a function $f(x)$ be such that

(*i*) *$f(x)$, $f'(x)$, $f''(x)$ ...... $f^{n-1}(x)$ are continuous in the closed interval $[a, a + h]$.*

(*ii*) *$f^n(x)$ exists in the open interval $(a, a + h)$; then there exists at least one number θ, between 0 and 1, such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h) \quad [0 < \theta < 1]$$

NOTES

Proof. Let $F(x) = f(x) + (a + h - x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x)$

$$+ \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \mathbf{(a + h - x)\,A} \quad \ldots(1)$$

where A is a constant such that F(a) = F($a + h$).

Putting x = a in (1),

$$F(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \quad \ldots(2)$$

Putting x = a + h in (1),

$$F(a + h) = f(a + h) + 0 + 0 + \ldots\ldots = f(a + h) \quad \ldots(3)$$

But F($a + h$) = F(a) (assumption)

Putting values from (3) and (2),

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \quad \ldots(4)$$

Now Let us apply Rolle's Theorem on F(x) in [a, a + h].

(*i*) F(x) is continuous in the closed interval [a, $a + h$].

[∵ $f(x), f'(x), f''(x) \ldots, f^{n-1}(x)$ are continuous in the closed interval [a, $a + h$] (given)
Also $(a+h-x), (a+h-x)^2 \ldots\ldots (a+h-x)^{n-1}$ being polynomials in x are continuous in [a, $a + h$]]

(*ii*) Similarly, F(x) is derivable in the open interval (a, $a + h$)

(*iii*) Also, F(a) = F($a + h$) [Assumption]

∴ By Rolle's Theorem, there exists at least one real number $\theta (0 < \theta < 1)$ such that $F'(a + \theta h) = 0$.

Differentiating both sides of (1) w.r.t. x,

$$F'(x) = f'(x) + [(a + h - x) f''(x) - f'(x)] + \frac{1}{2!} [(a + h - x)^2 f'''(x) - 2(a + h - x) f''(x)] \ldots$$

$$+ \frac{1}{(n-1)!} [(a + h - x)^{n-1} f^n(x) - (n - 1)(a + h - x)^{n-2} f^{n-1}(x)] - A$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{d}{dx}(v) + \frac{du}{dx}.\, v \text{ and } \frac{d}{dx}(a + h - x) = -1\right]$$

or $\quad F'(x) = \frac{1}{(n-1)!} (a + h - x)^{n-1} f^n(x) - A.$

Put $\quad x = a + \theta h,$

$$F'(a + \theta h) = \frac{1}{(n-1)!} (a + h - a - \theta h)^{n-1} f^n(a + \theta h) - A$$

$$= \frac{1}{(n-1)!} (h - \theta h)^{n-1} f^n(a + \theta h) - A$$

$$= \frac{h^{n-1}}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h) - A.$$

But $\quad F'(a + \theta h) = 0$

$$\therefore \quad \frac{h^{n-1}}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h) - A = 0$$

or $$A = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a+\theta h)$$

Putting this value of A in (4),

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h \cdot \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a+\theta h)$$

or $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{n}(a+\theta h)$$

NOTES

Note. F(x) of Art. 3 has been written from

F(x) of Art. 2 replacing last term $\frac{(a+h-x)^n}{n!}$ A by $(a + h - x)$ A.

Cor. Maclaurin's theorem with Cauchy's form of remainder after n terms.

Put $a = 0$ and $h = x$ in Art. 3. (Taylor's theorem with Cauchy form of remainder, then we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{n}(\theta x) \quad [0 < \theta < 1]$$

which is called **Maclaurin's theorem with Cauchy's form of remainder** and holds good when

(i) $f(x), f'(x), f''(x) \ldots\ldots f^{n-1}(x)$ *are all continuous in the closed interval* $[0, x]$

and *(ii)* $f^n(x)$ *exists in the open interval* $(0, x)$.

Note. The $(n + 1)$th term $\frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ $[0 < \theta < 1]$ is known as *Cauchy remainder after n terms* in Maclaurin's development of $f(x)$.

Example 1. *Expand a^x by Maclaurin's theorem with Lagrange's form of remainder after n terms.*

Sol. Here $f(x) = a^x$.

By successive differentiation, we get

$$f'(x) = a^x \cdot \log a,\ f''(x) = a^x \cdot (\log a)^2,\ \ldots\ldots,\ f^{n-1}(x) = a^x \cdot (\log a)^{n-1}$$

and $$f^n(x) = a^x \cdot (\log a)^n$$

Putting $x = 0$, we have

$$f(0) = 1,\ f'(0) = \log a,\ f''(0) = (\log a)^2,\ \ldots\ldots\ f^{n-1}(0) = (\log a)^{n-1} \quad [\because\ a^0 = 1]$$

and $$f^n(\theta x) = a^{\theta x} (\log a)^n$$

But by Maclaurin's theorem with Lagrange's form of remainder, we have

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x) \quad [0 < \theta < 1]$$

NOTES

Substituting the values in the above result, we get

$$a^x = 1 + x \,.\, \log a + \frac{x^2}{2!}(\log a)^2 + \ldots\ldots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}\, a^{\theta x} \,.\, (\log a)^n.$$

Here Lagrange's remainder after n terms $= \frac{x^n}{n!}\, a^{\theta x} \,.\, (\log a)^n$, where $0 < \theta < 1$.

Cor. Putting $a = e$, the expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\, e^{\theta x}\ [0 < \theta < 1].$$

Example 2. *Expand* $e^{ax} \sin bx$ *by Maclaurin's Theorem with Lagrange's form of remainder after* n *terms.*

Sol. Let $f(x) = e^{ax} \sin bx$

$\therefore$ $f'(x) = e^{ax} \cos bx \,.\, b + \sin bx \,.\, e^{ax} \,.\, a = e^{ax}(b \cos bx + a \sin bx)$

$f''(x) = e^{ax}(-b^2 \sin bx + ab \cos bx) + (b \cos bx + a \sin bx)\, e^{ax} \,.\, a$

$= e^{ax}[-b^2 \sin bx + ab \cos bx + ab \cos bx + a^2 \sin bx]$

or $f''(x) = e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$

$\therefore$ $f'''(x) = e^{ax}[(a^2 - b^2) \cos bx \,.\, b - 2ab^2 \sin bx]$

$+ [(a^2 - b^2) \sin bx + 2ab \cos bx]\, e^{ax} \,.\, a$

....

and we know that $f^n(x) = (a^2 + b^2)^{n/2}\, e^{ax} \sin\left(bx + n \tan^{-1} \frac{b}{a}\right)$

Putting $x = 0$ in $f(x), f'(x), f''(x), f'''(x)$ and changing x to θx in $f^n(x)$

we have $f(0) = 0,\ f'(0) = b,\ f''(0) = 2ab,$

$f'''(0) = (a^2 - b^2)\, b + (2ab)a = a^2 b - b^3 + 2a^2 b = 3a^2 b - b^3 = b(3a^2 - b^2)$

and $f^n(\theta x) = (a^2 + b^2)^{n/2}\, e^{a\theta x} \sin\left(b\theta x + n \tan^{-1} \frac{b}{a}\right).$

Putting these values in Maclaurin's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(\theta x),$$

we have $e^{ax} \sin bx = bx + \frac{x^2}{2!}(2ab) + \frac{x^3}{3!}\, b(3a^2 - b^2) + \ldots\ldots$

$$\ldots\ldots + \frac{x^n}{n!}(a^2 + b^2)^{n/2}\, e^{a\theta x} \sin\left(b\theta x + n \tan^{-1} \frac{b}{a}\right).$$

EXERCISE 1

1. (*a*) State and prove Taylor's theorem with Lagrange's form of remainder after n terms.
   (*b*) State and prove Taylor's development of a function with Cauchy's form of remainder and hence deduce the Maclaurin's expansion.
   (*c*) State and prove Maclaurin's theorem (As special case of Taylor's theorem).
   (*d*) State and prove Maclaurin's theorem with Lagrange's form of remainder.
   [**Hint.** Cor. Art. 2.]

NOTES

2. Show that

(*i*) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n}$.

$\left[\textbf{Hint. } \frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}.\right]$

(*ii*) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots. - \frac{x^{n-1}}{n-1} - \frac{x^n}{n(1-\theta x)^n}$.

3. Show that $e^{ax}\cos bx = 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + \frac{x^3}{3!}a(a^2 - 3b^2) + \ldots\ldots$

$$+ \frac{x^n}{n!}(a^2+b^2)^{n/2} e^{a\theta x}\cos\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$$

$\left[\textbf{Hint. } \frac{d^n}{dx^n}(e^{ax}\cos bx) = (a^2+b^2)^{n/2} e^{ax}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right).\right]$

4. Show that for every value of x,

(*i*) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n}}{(2n)!}\sin(\theta x)$.

$\left[\textbf{Hint. } \frac{d^n}{dx^n}\sin(ax+b) = a^n\sin\left(ax+b+\frac{n\pi}{2}\right) \text{ Also } \sin(n\pi+\theta) = (-1)^n\sin\theta.\right]$

(*ii*) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin(\theta x)$

$\left[\textbf{Hint. } \frac{d^n}{dx^n}\cos(ax+b) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right) \text{ and } \cos(n\pi+\theta) = (-1)^n\cos\theta.\right]$

5. Show that $\sin x + \cos x = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4(\sin\theta x + \cos\theta x)$.

6. Show that $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \ldots\ldots + (-1)^{n-1}\frac{h^n}{n(x+\theta h)^n}$.

[**Hint.** Use Taylor's Theorem with Lagrange's form of Remainder.]

7. Expand $e^{ax}\sin px$ by Maclaurin's Theorem with Cauchy's form of remainder after n terms.

$\left[\textbf{Hint. } \frac{d^n}{dx^n} e^{ax}\sin bx = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right)\right]$

8. Expand $b^x\cos ax$ by Maclaurin's Theorem with Lagrange's form of remainder after r terms.

[**Hint.** $b^x = e^{\log b^x}$ $(\because\ e^{\log f(x)} = f(x)) = e^{x\log b}$ $\therefore$ $b^x\cos ax = e^{x\log b}\cos ax$.]

Answers

7. $px + \frac{x^2}{2!}(2ap) + \frac{x^3}{3!}p(3a^2 - p^2) + \ldots\ldots$

$$+ x^n\frac{(1-\theta)^{n-1}}{(n-1)!}e^{a\theta x}(a^2+p^2)^{n/2}\sin\left(p\theta x + n\tan^{-1}\frac{p}{a}\right).$$

8. $1 + x\log b + \frac{x^2}{2!}[(\log b)^2 - a^2] + \frac{x^3}{3!}[(\log b)^2 - 3a^2]\log b$

$$+ \ldots\ldots + \frac{x^r}{r!}[(\log b)^2 + a^2]^{r/2} b^{\theta x}\cos\left(a\theta x + r\tan^{-1}\frac{a}{\log b}\right).$$

NOTES

3. EXPANSIONS

STRUCTURE

- Introduction
- Convergent and Divergent Series
- Failure of Taylor's and Maclaurin's Theorem
- Application of Maclaurin's Theorem
- Application of Taylor's Theorem
- Another Form of Taylor's Series
- Method of Differential Equation

INTRODUCTION

A set of terms formed and arranged according to some *definite law* and connected by the signs + ve or – ve is called a **series.** There are two kinds of series ; **finite** and **infinite.** A series is said to be *finite* when the number of terms it contains is limited and definite. For example, 2 + 4 + 6 + 8 + 10 is finite series of 5 terms whose sum is 30. The sum of a finite series is the sum of all the terms it is made up of.

Infinite Series. A series is said to be infinite when the number of terms it contains is unlimited *i.e.,* infinitely large so that it has no last term. For example, $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$ to infinity, it is an infinite series each term of which is equal to one-third of the term preceding it. It is an infinite G.P.

CONVERGENT AND DIVERGENT SERIES

If S_n the sum of first n terms of an infinite series tends to a definite limit S, as n approaches infinity, the infinite series is said to be convergent and this limit S is called the sum of the infinite series.

Illustration. The infinite series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$ is convergent.

Proof. Let S_n denotes the sum to n terms

then $$S_n = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ... \text{ to } n \text{ terms}$$

$$= \frac{1\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{3}{2} - \frac{1}{2} \cdot \frac{1}{3^{n-1}}$$

Now if limit of S_n when $n \to \infty$, be denoted by S, then

$$S = \frac{3}{2} \qquad \left(\because \text{ as } n \to \infty, \frac{1}{3^{n-1}} \to 0 \right)$$

Since the limit of S_n when $n \to \infty$ is definite and finite ; the series is convergent.

The second kind of the infinite series in which S_n, the sum of first n terms does not tend to a finite limit but increases beyond all bounds as n increases indefinitely, is called a **Divergent Series.** It is evident that the question of the sum of a non-convergents series does not arise.

Illustration 1. The series $1 + 2 + 3 + 4 + \ldots\ldots \infty$ is a divergent series.

Here $$S_n = \frac{n(n+1)}{2} \quad \text{and} \quad \underset{n \to \infty}{\text{Lt}} S_n = \underset{n \to \infty}{\text{Lt}} \frac{n(n+1)}{2} = \infty$$

Since the limit of S_n when $n \to \infty$ is not definite and finite, the series is divergent.

Note. If we do not take into consideration convergency and divergency of series we may be led to erroneous conclusions as the next illustration will show.

Illustration 2. $\frac{1}{1-x}$ is a function of x and we can represent it by an infinite series :

By actual division, we have $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots + x^n + \ldots \qquad \ldots(1)$$

an equality which may be expected to hold good for all values of x.

Putting $x = 3$, we get $$\frac{1}{1-3} = 1 + 3 + 3^2 + 3^3 + \ldots\ldots$$

or $$-\tfrac{1}{2} = 1 + 3 + 3^2 + 3^3 + \ldots\ldots$$

This is an absurd result for a small negative quantity is equal to a very large positive quantity.

The absurdity is due to the fact that the infinite series on the right hand side of (1) which is a G.P. is convergent only if x lies between – 1 and 1, and therefore we are not justified in putting $x = 3$ on both sides of (1).

3. Taylor's Infinite Series

If a function $f(x)$ *possesses derivatives of all orders* in the interval $(a, a + h)$, then for every integer n however large, there corresponds a Taylor's development with Lagrange's form of remainder namely

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where $$R_n = \frac{h^n}{n!} f^n(a + \theta h) \qquad [0 < \theta < 1]$$

or $$f(a+h) = S_n + R_n$$

where $$S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

Now, suppose $R_n \to 0$, as $n \to \infty$, then

$$\underset{n \to \infty}{\text{Lt}} S_n = f(a+h)$$

or $$f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \ldots\ldots$$

converges and its sum is equal to $f(a + h)$.

NOTES

This leads to

Theorem. *If $f(x)$ be a function*

(*i*) *possessing derivatives of* **all orders** *in the interval $(a, a + h)$ and*

(*ii*) *Taylor's remainder* $R_n = \frac{h^n}{n!} f^n(a + \theta h)$ *tends to zero as $n \to \infty$, then*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^n}{n!} f^n(a) + \ldots + \text{ad inf.} \quad \ldots(A)$$

This theorem expands $f(a + h)$ in an infinite series of ascending integral powers of h and the series is called **Taylor's infinite series.**

Other Forms of Taylor's Infinite Series :

(*i*) Writing x for a, in (A), we have

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots + \frac{h^n}{n!} f^n(x) + \ldots\ldots$$

(*ii*) Putting $a + h = b$ or $h = b - a$, in (A), we get

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^n}{n!} f^n(a) + \ldots\ldots$$

(*iii*) Changing $a + h$ to x *i.e.,* h to $x - a$, in (A), we have

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots\ldots$$

which expands $f(x)$ in ascending integral powers of $(x - a)$.

Cor. Maclaurin's Infinite Series.

When $a = 0$ and $h = x$ in the above theorem, we notice that if

(*i*) *$f(x)$ possesses derivative of* **all orders** *in the interval $(0, x)$ and*

(*ii*) *Maclaurin's remainder* $R_n = \frac{x^n}{n!} f^n(\theta x)$ *tends to zero as $n \to \infty$, then*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots \text{ ad. inf.}$$

which is known as Maclaurin's infinite series.

Note. If the function $f(x)$ is denoted by y, then the expansion may be written in the form

$$y = y(0) + x \,.\, y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots\ldots + \frac{x^n}{n!} y_n(0) + \ldots\ldots$$

where $y(0), y_1(0), y_2(0), \ldots\ldots, y_n(0)$ etc. denote values of $y, y_1, y_2, \ldots\ldots, y_n$ respectively for $x = 0$.

FAILURE OF TAYLOR'S AND MACLAURIN'S THEOREM

(*a*) Taylor's theorem *fails* to expand $f(x + h)$ in an infinite series in the following situations :

(*i*) if any of functions $f(x), f'(x), f''(x)$ .... becomes infinite or does not exist for any value of x in the interval under consideration,

or (*ii*) if R_n does not tend to zero as $n \to \infty$.

(*b*) Maclaurin's theorem *fails* to expand $f(x)$ in an infinite series in the following situations :

(*i*) if any of $f(x), f'(x), f''(x)$ ...... becomes infinite or does not exist in the closed interval $[0, x]$,

or (*ii*) if R_n does not tend to zero as $n \to \infty$.

Note. We observe that before we can expand a given function as an infinite Taylor's or Maclaurin's series 2 it is essential to examine the behaviour of R_n as n tends to infinity. But this is not simple in many cases. We, therefore, generally obtain the expansion by *assuming* the possibility of expanding it in an infinite series by assuming that R_n tends to 0 as n tends to infinity.

APPLICATION OF MACLAURIN'S THEOREM

Working Rule

1st Step. Put the given function equal to $f(x)$.

2nd Step. Differentiate $f(x)$, a number of times and find $f'(x)$, $f''(x)$, $f'''(x)$ .... and so on.

3rd Step. Put $x = 0$ in the results obtained in 2nd Step and find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ ...... and so on.

4th Step. Now substitute the values of $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, ... in,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ...$$

Example 1. *Expand a^x and e^x in powers of x, by Maclaurin's theorem.*

Sol. Let $f(x) = a^x$ $\therefore$ $f(0) = a^0 = 1$

then $f'(x) = a^x \log a$ $\quad$ $f'(0) = \log a$

$f''(x) = a^x (\log a)^2$ $\quad$ $f''(0) = (\log a)^2$

$f'''(x) = a^x (\log a)^3$ $\quad$ $f'''(0) = (\log a)^3$

and so on ....

Now Maclaurin's expansion is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ...$

Putting values of $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, we have

$$\mathbf{a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!} (\log a)^3 + ...} \qquad ...(1)$$

Putting $a = e$, we have $\mathbf{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...}$ $\qquad ...(2)$

Note. The series (1) and (2) are called exponential series and are convergent for all values of x.

Example 2. *Expand $\log(1 + x)$ in powers of x.*

Sol. Let $f(x) = \log(1 + x)$ $\therefore$ $f(0) = \log 1 = 0$

then $f'(x) = \frac{1}{1+x} = (1 + x)^{-1}$ $\therefore$ $f'(0) = 1$

$f''(x) = (-1)(1 + x)^{-2}$ $\therefore$ $f''(0) = -1$

$f'''(x) = (-1)(-2)(1 + x)^{-3}$ $\therefore$ $f'''(0) = 2$

$f^{iv}(x) = (-1)(-2)(-3)(1 + x)^{-4}$ $\therefore$ $f^{iv}(0) = -2.3 = -6$

and so on.

Putting these values of $f(0)$, $f'(0)$, $f''(0)$ etc. in Maclaurin's expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + ...,$$

NOTES

we have $\log(1+x) = 0 + x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots$

or $$\mathbf{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots}$$

Changing x into $-x$, we have

$$\mathbf{\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots}$$

Cor. (*i*) $\log(1+x) - \log(1-x)$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\right]$$

$\therefore$ $$\mathbf{\log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots\right]}$$

Cor. (*ii*) $\log(1+x)^m = m\log(1+x) = m\left[x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots\right]$

Note. The above series for $\log(1+x)$ is called logarithmic series and is convergent *i.e.,* valid for $|x|$ less than 1 and also for $x = 1$.

Example 3. *Expand sin x and cos x in powers of x and hence find cos 18°.*

Sol. Let $f(x) = \sin x$ $\therefore$ $f(0) = 0$

then $f'(x) = \cos x$ $\therefore$ $f'(0) = 1$

$f''(x) = -\sin x$ $\therefore$ $f''(0) = 0$

$f'''(x) = -\cos x$ $\therefore$ $f'''(0) = -1$

$f^{iv}(x) = \sin x$ $\therefore$ $f^{iv}(0) = 0$

$f^{v}(x) = \cos x$ $\therefore$ $f^{v}(0) = 1$

and so on, ...

Putting these values of $f(0), f'(0), f''(0), f'''(0)$ in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots,$$

we have $$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) - \ldots\ldots$$

or $$\mathbf{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots} \qquad \ldots(1)$$

In a like manner, we get $\mathbf{\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots}$ $\qquad\ldots(2)$

Note. Series (1) and (2) in Example 3 are convergent for all values of x.

To find cos 18°.

Now $$x = 18° = \frac{\pi}{10} = \frac{3.14}{10} = 0.314$$

Putting $x = \dfrac{\pi}{10}$ in (2), we get

$$\cos 18° = \cos\frac{\pi}{10} = 1 - \left(\frac{\pi}{10}\right)^2 \cdot \frac{1}{2!} + \left(\frac{\pi}{10}\right)^4 \frac{1}{4!} - \ldots\ldots$$

$$= 1 - \frac{(0.314)^2}{2!} + \frac{(0.314)^4}{4!} - \ldots\ldots \qquad \left[\because \frac{\pi}{10} = .314 \text{ approx.}\right]$$

$$= 1 - 0.04929 + 0.00040 = 0.95111 = 0.9511 \text{ nearly.}$$

NOTES

Example 4. *Expand $\sin^{-1} x$ upto four terms in powers of x.*

Sol. Let $f(x) = \sin^{-1} x \qquad \therefore \quad f(0) = 0$

Also $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1 - x^2)^{-1/2} \qquad \therefore \quad f'(0) = 1$

Expanding by Binomial theorem,

$$f'(x) = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x^2)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \ldots..$$

$$\left[\because \text{ By Binomial theorem } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\right]$$

or $$f'(x) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{15}{48}x^6 + \ldots\ldots$$

$$f''(x) = x + \frac{3}{2}x^3 + \frac{15}{8}x^5 + \ldots\ldots \qquad \therefore \qquad f''(0) = 0$$

$$f'''(x) = 1 + \frac{9}{2}x^2 + \frac{75}{8}x^4 + \ldots\ldots \qquad \therefore \qquad f'''(0) = 1$$

$$f^{iv}(x) = 9x + \frac{75}{2}x^3 + \ldots\ldots \qquad f^{iv}(0) = 0$$

$$f^{v}(x) = 9 + \frac{225}{2}x^2 + \ldots\ldots \qquad f^{v}(0) = 9$$

$$f^{vi}(x) = 225x + \ldots\ldots \qquad f^{vi}(0) = 0$$

and proceeding further, $f^{vii}(0) = 225$, etc.

Putting these values in Maclaurin's expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots,$$

we have $$\sin^{-1} x = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(9) + \frac{x^6}{6!}(0) + \frac{x^7}{7!}(225) + \ldots\ldots$$

$$= x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \frac{225x^7}{7!} + \ldots\ldots$$

$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} + \ldots\ldots$$

Note. To expand an alone inverse function ; find its first derivative. *Expand by Binomial Theorem* and then find other derivatives.

Example 5. *Expand $\tan^{-1} x$.*

Sol. Let $f(x) = \tan^{-1} x, \qquad \therefore \quad f(0) = 0$

then $f'(x) = \frac{1}{1+x^2} = (1 + x^2)^{-1} \qquad \therefore \quad f'(0) = 1$

$$= 1 - x^2 + x^4 - x^6 + \ldots\ldots \qquad \text{[By Binomial Theorem]}$$

$$f''(x) = -2x + 4x^3 - 6x^5 + \ldots\ldots \qquad f''(0) = 0$$
$$f'''(x) = -2 + 12x^2 - 30x^4 + \ldots\ldots \qquad f'''(0) = -2$$
$$f^{iv}(x) = 24x - 120x^3 + \ldots\ldots \qquad f^{iv}(0) = 0$$
$$f^{v}(x) = 24 - 360x^2 + \ldots\ldots \qquad f^{v}(0) = 24$$

NOTES

and so on ....

Putting these values of $f(0), f'(0), f''(0)$ etc. in Maclaurin's expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots, \text{ we have}$$

$$\tan^{-1} x = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24) + \ldots\ldots$$

or $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

Note 1. The expansion for $\tan^{-1} x$ is valid only if $-1 < x \leq 1$.

2. This expansion for $\tan^{-1} x$ known as **Gregory's series** is very useful for finding the value of π.

Example 6. *Expand tan x by Maclaurin's Theorem as far as x^5 and hence find the value of tan 46° 30′ upto four decimal places.*

Sol. Let $f(x) = \tan x \qquad \therefore \quad f(0) = 0$

then $f'(x) = \sec^2 x = 1 + \tan^2 x$ **Note this step** $\qquad \therefore \quad f'(0) = 1$

$$f''(x) = 2 \tan x \sec^2 x = 2 \tan x (1 + \tan^2 x)$$
$$= 2 \tan x + 2 \tan^3 x \qquad \therefore \quad f''(0) = 0$$
$$f'''(x) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x)$$
$$= 2 + 8 \tan^2 x + 6 \tan^4 x \qquad \therefore \quad f'''(0) = 2$$
$$f^{iv}(x) = 16 \tan x \sec^2 x + 24 \tan^3 x \sec^2 x$$
$$= 8 \sec^2 x(2 \tan x + 3 \tan^3 x)$$
$$= 8(1 + \tan^2 x)(2 \tan x + 3 \tan^3 x)$$
$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x \qquad \therefore \quad f^{iv}(0) = 0$$
$$f^{v}(x) = 16 \sec^2 x + 120 \tan^2 x \sec^2 x + 120 \tan^4 x \sec^2 x$$
$$= 8 \sec^2 x (2 + 15 \tan^2 x + 15 \tan^4 x) \qquad \therefore \quad f^{v}(0) = 16$$

Putting these values in Maclaurin's expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \ldots\ldots$$

we have $$\tan x = 0 + x + \frac{x^3}{3!} \cdot 2 + \frac{x^5}{5!} \cdot 16 \ldots\ldots$$

or $$\mathbf{\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots\ldots} \qquad \ldots (1)$$

Deduction. Here $x = 46° \, 30' = \left(46 \frac{1}{2}\right)^\circ$ $\qquad | \because \quad 1° = 60' \quad \therefore \quad 30' = \left(\frac{1}{2}\right)^\circ$

$$= \left(\frac{93}{2}\right)^\circ = \frac{93}{2} \times \frac{\pi}{180} \text{ Radians}$$

$$= \frac{31}{120} \times \frac{22}{7} = \frac{31 \times 11}{60 \times 7} = \frac{341}{420} = 0.812$$

Putting $x = 46° \, 30' = 0.812$ in eqn. (1), we have

NOTES

$$\tan 46° \, 30' = 0.812 + \frac{(0.812)^3}{3} + \frac{2}{15} (0.812)^5$$

$$= 0.812 + \frac{0.5353}{3} + \frac{2}{15} (0.3530)$$

$$= 0.812 + 0.1784 + 0.047 = \mathbf{1.0374.}$$

Note. The result of above eqn. (1) can be used as a formula also.

Example 7. *Expand by Maclaurin's Theorem* $\dfrac{e^x}{e^x + 1}$, *as far as* x^3.

Sol. Let $f(x) = \dfrac{e^x}{e^x + 1}$ $\quad \therefore \quad f(0) = \dfrac{e^0}{e^0 + 1} = \dfrac{1}{2}$

then $f'(x) = \dfrac{(e^x + 1)\, e^x - e^x \,.\, e^x}{(e^x + 1)^2} = \dfrac{e^x}{(e^x + 1)^2}$ $\quad \therefore \quad f'(0) = \dfrac{e^0}{(e^0 + 1)^2} = \dfrac{1}{2^2}$

$$f''(x) = \frac{(e^x + 1)^2\, e^x - e^x\, 2(e^x + 1)e^x}{(e^x + 1)^4} = \frac{(e^x + 1)\, e^x - 2e^{2x}}{(e^x + 1)^3}$$

$$= \frac{e^{2x} + e^x - 2e^{2x}}{(e^x + 1)^3} = \frac{e^x - e^{2x}}{(e^x + 1)^3} \quad \therefore \quad f''(0) = \frac{e^0 - e^0}{(e^0 + 1)^3} = 0$$

$$f'''(x) = \frac{(e^x + 1)^3\, (e^x - 2e^{2x}) - (e^x - e^{2x})\,.\,3\,(e^x + 1)^2\, e^x}{(e^x + 1)^6}$$

$$= \frac{(e^x + 1)^2\, [(e^x + 1)(e^x - 2e^{2x}) - 3e^x (e^x - e^{2x})]}{(e^x + 1)^6}$$

$$= \frac{e^x - 4e^{2x} + e^{3x}}{(e^x + 1)^4} \quad \therefore \quad f'''(0) = -\frac{1}{2^3}$$

Putting these values of $f(0),\ f'(0),\ f''(0),\ f'''(0)$ in Maclaurin's expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

We have $\dfrac{e^x}{e^x + 1} = \dfrac{1}{2} + \dfrac{x}{4} - \dfrac{x^3}{48} + \ldots\ldots$

Example 8. *Prove by Maclaurin's Theorem that*

$$e^{\sin x} = 1 + x + \frac{x^2}{1.2} - \frac{3.x^4}{1.2.3.4} + \ldots\ldots$$

Sol. Let $f(x) = e^{\sin x}$ $\quad \therefore \quad f(0) = e^0 = 1$

$f'(x) = e^{\sin x} \cos x$ $\quad \therefore \quad f'(0) = e^0 \cos 0 = 1$

$f''(x) = e^{\sin x} (-\sin x) + \cos x \, e^{\sin x} \cos x$

$= e^{\sin x} [\cos^2 x - \sin x]$ $\quad \therefore \quad f''(0) = e^0 [1 - 0] = 1$

$f'''(x) = e^{\sin x} [2 \cos x(-\sin x) - \cos x] + e^{\sin x} \cos x \,.\, [\cos^2 x - \sin x]$

$= e^{\sin x} \cos x \,.\, [-2 \sin x - 1 + \cos^2 x - \sin x]$

$= -e^{\sin x} \cos x[3 \sin x + \sin^2 x]$ $\quad \therefore \quad f'''(0) = 0$

NOTES

$$f^{iv}(x) = -e^{\sin x}\cos x[3\cos x + 2\sin x\cos x] + e^{\sin x}\,.\,\sin x[3\sin x + \sin^2 x] - [3\sin x + \sin^2 x]\cos x\, e^{\sin x}\,.\,\cos x$$

$$\left[\because \frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}\right]$$

$$\therefore \quad f^{iv}(0) = -3$$

Putting these values in Maclaurin's Theorem, namely

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots,$$

we have $$e^{\sin x} = 1 + x.1 + \frac{x^2}{2!}\,.\,1 + \frac{x^3}{3!}\,.\,0 + \frac{x^4}{4!}(-3) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{1.2} - \frac{3x^4}{1.2.3.4} + \ldots\ldots$$

Example 9. *Expand $e^{ax}\cos bx$, as an infinite series of ascending powers of x. Give also the (n + 1)th term of the above series.*

Sol. Let $f(x) = e^{ax}\cos bx,$

$$f'(x) = a\,.\,e^{ax}\cos bx - e^{ax}\sin bx\,.\,b = e^{ax}(a\cos bx - b\sin bx)$$

$$f''(x) = ae^{ax}(a\cos bx - b\sin bx) + e^{ax}(-ab\sin bx - b^2\cos bx)$$

$$= e^{ax}[(a^2 - b^2)\cos bx - 2ab\sin bx]$$

$$f'''(x) = ae^{ax}[(a^2 - b^2)\cos bx - 2ab\sin bx] + e^{ax}[(a^2 - b^2)(-b\sin bx) - 2ab^2\cos bx]$$

$$= e^{ax}[a(a^2 - b^2)\cos bx - 2a^2b\sin bx - b(a^2 - b^2)\sin bx - 2ab^2\cos bx]$$

$$= e^{ax}[a^3\cos bx + b^3\sin bx - 3ab^2\cos bx - 3a^2b\sin bx]$$

....

$$f^n(x) = (a^2 + b^2)^{n/2}e^{ax}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right)$$

Putting $x = 0$ in the above

$$f(0) = 1,\; f'(0) = a,\; f''(0) = a^2 - b^2$$

$$f'''(0) = a(a^2 - 3b^2), \ldots\ldots\ldots\; f^n(0) = (a^2 + b^2)^{n/2}\cos\left(n\tan^{-1}\frac{b}{a}\right)$$

Putting these values of $f(0)$, $f'(0)$ etc. in Maclaurin's expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots,$$

we have $$e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2!}x^2 + \frac{a(a^2 - 3b^2)}{3!}x^3 + \ldots\ldots$$

$$\ldots\ldots + \frac{(a^2 + b^2)^{n/2}}{n!}x^n\cos\left(n\tan^{-1}\frac{b}{a}\right) + \ldots\ldots$$

Example 10. *Expand $\sin(e^x - 1)$ upto and including the term x^4.*

Sol. Let $f(x) = \sin(e^x - 1).$

Put $x = 0$, $f(0) = \sin(e^0 - 1) = \sin 0 = 0$

$\therefore$ $f'(x) = \cos(e^x - 1)\,.\,e^x = e^x\cos(e^x - 1)$

Put $x = 0$, $f'(0) = e^0\cos(1 - 1) = 1\cos 0 = 1.$

Again diff. w.r.t. x,

$$f''(x) = e^x[-\sin(e^x - 1)]\, e^x + \cos(e^x - 1) \,.\, e^x$$

or $$f''(x) = -e^{2x}\sin(e^x - 1) + e^x\cos(e^x - 1)$$

Put $x = 0$, $f''(0) = 0 + 1\cos 0 = 1$.

Again diff. w.r.t. x,

$$f'''(x) = -[e^{2x}\cos(e^x - 1)\,.\,e^x + \sin(e^x - 1)\,.\,e^{2x}\,.\,2] + e^x[-\sin(e^x - 1)\,.\,e^x] + \cos(e^x - 1)\,.\,e^x$$

or $$f'''(x) = -e^{3x}\cos(e^x - 1) - 3e^{2x}\sin(e^x - 1) + e^x\cos(e^x - 1)$$

Put x = 0, $f'''(0) = -1 - 0 + 1 = 0$.

Again diff. w.r.t. x,

$$f^{iv}(x) = -[-e^{3x}\sin(e^x - 1)\,.\,e^x + \cos(e^x - 1)\,.\,e^{3x}\,.\,3] - 3[e^{2x}\cos(e^x - 1)\,.\,e^x + \sin(e^x - 1)\,.\,e^{2x}\,.\,2] - e^x\sin(e^x - 1)\,.\,e^x + \cos(e^x - 1)\,.\,e^x.$$

Put x = 0, $f^{iv}(0) = -3 - 3(1 + 0) - 0 + 1 = -5$

Putting these values of $f(0), f'(0), f''(0), f'''(0), f^{iv}(0)$ in Maclaurin's expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots,$$

We have $$\sin(e^x - 1) = 0 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-5) + \ldots\ldots$$

or $$\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots\ldots$$

Example 11. *Show that* $\log\dfrac{\tan x}{x} = \dfrac{x^2}{3} + \dfrac{7}{90}x^4 + \ldots\ldots$

Sol. L.H.S. $= \log\dfrac{\tan x}{x}$

We know that

$$\mathbf{\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots} \qquad \text{(By Eqn. (1) Example 6)}$$

$$\therefore \quad \text{L.H.S.} = \log\left(\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots}{x}\right) = \log\left[1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right]$$

$$= \log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots \text{ where } z = \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\ldots$$

Putting $z = \dfrac{x^2}{3} + \dfrac{2}{15}x^4 + \ldots\ldots$, we get

$$\text{L.H.S.} = \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots..\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^2 + \frac{1}{3}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^3$$

$$= \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots - \frac{1}{2}\left(\frac{x^4}{9} + \ldots\right) = \frac{x^2}{3} + \frac{2}{15}x^4 - \frac{x^4}{18} - \ldots\ldots$$

$$= \frac{x^2}{3} + \frac{7x^4}{90} + \ldots\ldots = \text{R.H.S.}$$

NOTES

EXERCISE 1

Apply Maclaurin's Theorem to prove (1 – 4)

1. (*a*) $\log \sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \ldots\ldots$ (*b*) $\log \cos x = -\dfrac{x^2}{2!} - 2.\dfrac{x^4}{4!} - 16.\dfrac{x^6}{6!} - \ldots..$

2. (*a*) $\sec x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \ldots\ldots$ (*b*) $\cos^2 x = 1 - x^2 + \dfrac{x^4}{3} + \ldots\ldots$

3. $\tan^{-1}(1 + x) = \dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x^2}{4} + \dfrac{x^3}{12} + \ldots\ldots$

4. $e^{ax} \sin bx = bx + abx^2 + \dfrac{b(3a^2 - b^2)}{3!} x^3 + \ldots. + \dfrac{(a^2 + b^2)^{n/2}}{n!} x^n \sin\left(n \tan^{-1} \dfrac{b}{a}\right) + \ldots..$

5. (*a*) Show that $\tan^{-1} \dfrac{2x}{1 - x^2} = 2\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots..\right)$.

[**Hint.** Put $x = \tan \theta$.]

(*b*) Expand $\sin^{-1}\left(\dfrac{2x^2}{1 + x^4}\right)$ in series of ascending powers of x.

[**Hint.** Put $x^2 = \tan \theta$.]

(*c*) Show that $\tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x} = \dfrac{x}{2} - \dfrac{x^3}{6} + \dfrac{x^5}{10} - \dfrac{x^7}{14} + \ldots\ldots$

[**Hint.** Put $x = \tan \theta$.]

6. Use Maclaurin's Theorem to find the expansion in the ascending powers of x of $\log_e (1 + e^x)$ upto the term including x^4.

7. Use Maclaurin's theorem to prove that $\log (1 + \sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + \ldots\ldots$

Point out the case of failure, if any.

[**Hint.** See Note Example 2.]

8. Prove that $e^x \log (1 + x) = x + \dfrac{x^2}{2!} + \dfrac{2x^3}{3!} + \ldots\ldots$

9. (*a*) Apply Maclaurin's theorem to show that $e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2 x^4}{4!} - \dfrac{2^2 x^5}{5!} + \ldots\ldots$

(*b*) Prove that $e^x \sec x = 1 + x + 2.\dfrac{x^2}{2!} + 4.\dfrac{x^3}{3!} + \ldots..$

10. Expand by Maclaurin's theorem $\tan^2 x$ in ascending powers of x as far as x^4.

11. Expand $\log (1 + \sin^2 x)$ in powers of x as far as the term containing x^4.

12. (*a*) By Maclaurin's theorem expand $\log [1 - \log (1 - x)]$ in powers of x as far as the term in x^3.

(*b*) Expand by Maclaurin's series $\log [1 + \log (1 + x)]$ in powers of x as far as term containing x^3.

13. (*a*) By Maclaurin's theorem find the first four terms in the expansion of $\log (1 + \tan x)$.

(*b*) Prove that $\log (1 + \tan x) = x - \dfrac{x^2}{2} + \dfrac{2x^3}{3} - \dfrac{7x^4}{12} + \ldots\ldots$

14. Show that $\log (1 - x + x^2) = -x + \dfrac{x^2}{2} + \dfrac{2x^3}{3} + \dfrac{x^4}{4} - \dfrac{x^5}{5} - \dfrac{2x^6}{6} - \dfrac{x^7}{7} + \dfrac{x^8}{8} \ldots\ldots$

$\left[\textbf{Hint}.\ \log (1 - x + x^2) = \log\left(\dfrac{1 + x^3}{1 + x}\right) = \log (1 + x^3) - \log (1 + x).\right]$

Answers

5. (*b*) $2\left[x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - ......\right]$ **6.** $\log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ...$

7. Case of failure is $\sin x = -1$ **10.** $x^2 + \frac{2x^4}{3} + ...$

NOTES

11. $x^2 - \frac{5}{6}x^4 + ...$ **12.** (*a*) $x + \frac{x^3}{6} + ...$ (*b*) $x - x^2 + \frac{7}{6}x^3 + ...$

13. $x - \frac{x^2}{2} + \frac{2x^3}{3} - \frac{7x^4}{12} + ...$

APPLICATION OF TAYLOR'S THEOREM

Working Rule *can be stated as :*

Step 1. Put the given function equal to $f(x + h)$.

Step 2. Put $h = 0$ and write $f(x)$.

Step 3. Differentiate $f(x)$ a number of times and obtain $f'(x)$, $f''(x)$, $f'''(x)$, etc. ....

Step 4. Now substitute the values of $f(x)$, $f'(x)$, $f''(x)$, ... in Taylor's expansion

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...$$

Example 1. *Apply Taylor's Theorem to find the expansion of log sin (x + h).*

Sol. (*i*) Let $f(x + h) = \log \sin (x + h)$

(*ii*) Putting $h = 0$, we have $f(x) = \log \sin x$

(*iii*) $\therefore$ $f'(x) = \frac{\cos x}{\sin x} = \cot x$

$f''(x) = -\operatorname{cosec}^2 x$

$f'''(x) = -2 \operatorname{cosec} x(-\operatorname{cosec} x \cot x) = 2 \operatorname{cosec}^2 x \cot x.$

....

Substituting these values of $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ in Taylor's expansion,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...,\text{ we get}$$

$$\log \sin (x + h) = \log \sin x + h \cot x - \frac{h^2}{2!} (\operatorname{cosec}^2 x) + \frac{h^3}{3!} (2 \operatorname{cosec}^2 x \cot x) + ...$$

$$= \log \sin x + h \cot x - \frac{h^2}{2} \operatorname{cosec}^2 x + \frac{h^3}{3} \operatorname{cosec}^2 x \cot x + ...$$

Example 2. *Expand $\sin^{-1} (x + h)$ in powers of h.*

Sol. (*i*) Let $f(x + h) = \sin^{-1} (x + h)$

(*ii*) Putting $h = 0$, we have $f(x) = \sin^{-1} x$.

NOTES

(*iii*) $\therefore$ $$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) = \frac{x}{(1-x^2)^{3/2}}$$

$$f'''(x) = \frac{(1-x^2)^{3/2} - \frac{3}{2}(1-x^2)^{1/2}(-2x).x}{(1-x^2)^3}$$

$$= \frac{(1-x^2)^{1/2}(1-x^2+3x^2)}{(1-x^2)^3} = \frac{1+2x^2}{(1-x^2)^{5/2}}$$

Putting these values of $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ in Taylor's expansion,

(*iv*) $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$, we have

$$\sin^{-1}(x+h) = \sin^{-1}x + \frac{h}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}} \cdot \frac{h^2}{2!} + \frac{1+2x^2}{(1-x^2)^{5/2}} \cdot \frac{h^3}{3!} + ...$$

Example 3. (*a*) *Expand sin* $(x+y)$ *in powers of y and deduce that*

sin $(x+y) = \sin x \cos y + \sin y \cos x$.

(*b*) *Obtain the value of sin 31° correct to four places of decimals.*

Sol. (*a*) (*i*) Let $f(x+y) = \sin(x+y)$

(*ii*) Putting $y = 0$, we have $f(x) = \sin x$

(*iii*) $\therefore$ $f'(x) = \cos x$; $f''(x) = -\sin x$,

$f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$, ...

.........................., and $f^n(x) = \sin(x + n\pi/2)$

By Taylor's expansion,

$$f(x+y) = f(x) + yf'(x) + \frac{y^2}{2!}f''(x) + ... + \frac{y^n}{n!}f^n(x) + ...$$

Putting values of $f(x)$, $f'(x)$, ... $f^n(x)$, we have,

$$\sin(x+y) = \sin x + y\cos x - \frac{y^2}{2!}\sin x - \frac{y^3}{3!}\cos x$$

$$+ \frac{y^4}{4!}\sin x + ... + \frac{y^n}{n!}\sin\left(x + \frac{n\pi}{2}\right) + ... \quad ...(1)$$

$$= \sin x\left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - ......\right] + \cos x\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - ......\right]$$

$$= \sin x \cos y + \cos x \sin y.$$

(*b*) Put $x = 30° = \frac{\pi}{6}$ and $y = 1° = \frac{\pi}{180} = .0175$ radians in (1),

Then $$\sin 31° = \sin\frac{\pi}{6} + \frac{\pi}{180}\cos\frac{\pi}{6} - \left(\frac{\pi}{180}\right)^2 \frac{1}{2!}\sin\frac{\pi}{6} ......$$

$$= .5 + (.0175)(.866) - \frac{(.0175)^2}{2!}(.5) ......$$

$$= .5 + .0151550 - .0000765 = .5150785 = .5151,$$

correct to four places of decimals.

NOTES

Example 4. (*i*) *If f(x) = $x^3 + 8x^2 + 15x - 24$, calculate the value of $f(\frac{11}{10})$ by the application of Taylor's series.*

(*ii*) *$f(x) = x^3 - 2x + 5$, find the value of f(2.001) with the help of Taylor's Theorem. Find the approximate change in the value of f(x) when x changes from 2 to 2.001.*

Sol. (*i*) By Taylor's theorem,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots \qquad \ldots(1)$$

To find $f\left(\frac{11}{10}\right)$ *i.e.,* $f\left(1+\frac{1}{10}\right)$

Put $x = 1$ and $h = \frac{1}{10}$ in the series (1),

$$f\left(\frac{11}{10}\right) = f\left(1+\frac{1}{10}\right) = f(1) + \frac{1}{10} f'(1) + \frac{1}{10^2}\cdot\frac{1}{2!} f''(1) + \frac{1}{3!}\cdot\frac{1}{(10)^3} f'''(1) + \ldots\ldots \qquad \ldots(2)$$

Now $f(x) = x^3 + 8x^2 + 15x - 24$ $\quad\therefore\quad f(1) = 0$

$f'(x) = 3x^2 + 16x + 15$ $\quad\therefore\quad f'(1) = 34$

$f''(x) = 6x + 16$ $\quad\therefore\quad f''(1) = 22$

$f'''(x) = 6$ $\quad\therefore\quad f'''(1) = 6$

$f^{iv}(x) = 0$ $\quad\therefore\quad f^{iv}(1) = 0$

Substituting values of $f(1), f'(1), f''(1)$ ..... etc. in (2), we get

$$f\left(1+\frac{1}{10}\right) = 0 + \frac{1}{10}34 + \frac{11}{100} + \frac{1}{1000} = 3.4 + .11 + .001 = 3.511.$$

(*ii*) Here put $x = 2$ and $h = .001$ in Taylor's series

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots,$$

we have $$f(2.001) = f(2) + (.001) f'(2) + \frac{(.001)^2}{2!} f''(2) + \frac{(.001)^3}{3!} f'''(2) + \ldots\ldots \qquad \ldots(3)$$

Now $f(x) = x^3 - 2x + 5$ $\quad\therefore\quad f(2) = 9$

$f'(x) = 3x^2 - 2$ $\quad\therefore\quad f'(2) = 10$

$f''(x) = 6x$ $\quad f''(2) = 12$

$f'''(x) = 6$ $\quad f'''(2) = 6$

$f^{iv}(x) = 0$ $\quad f^{iv}(2) = 0$

Putting these values in (3),

$$f(2.001) = 9 + (.001)\ 10 + \frac{1}{2!}(.001)^2\ (12) + \frac{1}{3!}(.001)^3\ (6) + \ldots\ldots$$

$$= 9 + .01 + .000006 + .000000001 = 9.010006001 = 9.01 \text{ approximately}$$

$\therefore$ Approximate change in the value of $f(x)$ as x changes from 2 to 2.001 $= f(2.001) - f(2)$

$$= 9.01 - 9 = .01 \text{ approximately.}$$

EXERCISE 2

NOTES

Apply Taylor's Theorem to prove the following expansions :

1. (*a*) $a^{x+h} = a^x \left[1 + h \log a + \dfrac{h^2}{2!}(\log a)^2 + \dfrac{h^3}{3!}(\log a)^3 + ...\right]$

(*b*) $e^{x+h} = e^x + he^x + \dfrac{h^2}{2!}e^x + ...$ (*c*) $\dfrac{1}{x+h} = \dfrac{1}{x} - \dfrac{h}{x^2} + \dfrac{h^2}{x^3} - \dfrac{h^3}{x^4} + ...$

(*d*) $\tan(x + h) = \tan x + h \sec^2 x + h^2 \sec^2 x \tan x + \dfrac{h^3}{3}\sec^2 x(1 + 3\tan^2 x) + ...$

2. (*a*) $\cos\left(\dfrac{\pi}{4} + h\right) = \dfrac{1}{\sqrt{2}}\left(1 - h - \dfrac{h^2}{2!} + \dfrac{h^3}{3!} + ...\right)$

(*b*) $\sin\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1}{\sqrt{2}}\left(1 + \theta - \dfrac{\theta^2}{2!} - \dfrac{\theta^3}{3!} + ...\right)$

3. (*a*) $\sec^{-1}(x + h) = \sec^{-1} x + \dfrac{h}{x\sqrt{x^2 - 1}} - \dfrac{2x^2 - 1}{x^2(x^2 - 1)^{3/2}} \cdot \dfrac{h^2}{2!} + ...$

(*b*) $\tan^{-1}(x + h) = \tan^{-1} x + \dfrac{h}{(1 + x^2)} - \dfrac{xh^2}{(1 + x^2)^2}....$

4. Prove that $\dfrac{f(x + h) + f(x - h)}{2} = f(x) + \dfrac{h^2}{2!}f''(x) + \dfrac{h^4}{4!}f^{iv}(x) + ...$

5. (*a*) Expand a^x by Taylor's Theorem.

[**Hint.** First expand a^{x+h} in ascending powers of h as in Q. 1(*a*). Then put $x = 0$ and change h into x.]

(*b*) Expand $\sin x$ and $\cos x$ by Taylor's Theorem.

6. (*a*) Find the approximate change in the value of $5x^3 - 3x^2 + 7x - 8$ as x changes from 3 to 3.001.

(*b*) If $f(x) = x^3 - 6x^2 + 7$; find the value of $f\left(\dfrac{21}{20}\right)$ by Taylor's Theorem. Also find the change in the value of $f(x)$ when x changes from 2 to 2.1.

(*c*) If $f(x) = 3x^3 - 5x^2 + 7$, find the value of $f\left(\dfrac{21}{20}\right)$ by Taylor's theorem and find the change in the value of $f(x)$ when x changes from 2 to 2.1.

(*d*) If $f(x) = x^3 + 2x^2 - 5x + 11$, find the value of $f\left(\dfrac{9}{10}\right)$ with the help of Taylor's series for $f(x + h)$.

(*e*) If $f(x) = x^3 + 6x^2 + 9$, find the value of $f\left(\dfrac{11}{10}\right)$ by Taylor's theorem.

7. Calculate the approximate value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's expansion.

[**Hint.** Let $f(x) = \sqrt{x} \quad \therefore \quad f(10) = \sqrt{10}$

or $\sqrt{10} = f(10) = f(9 + 1) = f(a + h)$ where $a = 9$, $h = 1$

$$= f(a) + h f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + ...$$

$$= f(9) + f'(9) + \frac{1}{2!}f''(9) + \frac{1}{3!}f'''(9) + ...]$$

8. (*a*) Calculate the approximate value of $\sqrt{17}$ to four decimal places by taking first three terms of a Taylor's Expansion.

(*b*) Calculate the approximate value of $\sqrt{26}$ to three decimal places by Taylor's expansion.

NOTES

9. Prove that $f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{1}{2!} f''(x) - \left(\frac{x}{1+x}\right)^3 \frac{1}{3!} f'''(x) + \ldots\ldots$

$\left[\textbf{Hint.} \text{ Write } \frac{x^2}{1+x} = \frac{x^2+x-x}{1+x} = \frac{x(x+1)-x}{1+x} = x - \frac{x}{1+x} = x + h, \text{ where } h = -\frac{x}{1+x}.\right]$

Answers

5. (*a*) $a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \frac{x^3}{3!} (\log a)^3 + \ldots$

(*b*) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

6. (*a*) 0.124 (*b*) 1.5426 (*c*) 4.960375, 1.733

(*d*) 8.849 (*e*) 17.591

7. 3.1623 **8.** (*a*) 4.123 (*b*) 5.099.

ANOTHER FORM OF TAYLOR'S SERIES

In this form $f(x)$ is expressed as a series in ascending integral powers of $x - a$.

We have $f(x) = f(a + x - a)$. Let $x - a = h$.

$$\therefore \quad f(x) = f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots$$

Now replacing h by $x - a$, we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots$$

Example 1. *Expand sin x in ascending powers of* $\left(x - \frac{\pi}{2}\right)$.

Sol. Here $f(x) = \sin x$

We know that $f(x) = f\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right)$

$\left[\text{We have done this step to get } \left(x - \frac{\pi}{2}\right)\right]$

$$= f(a+h), \text{ where } a = \frac{\pi}{2} \text{ and } h = x - \frac{\pi}{2}$$

i.e.,
$$f(x) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f^{iv}(a) + \ldots\ldots$$

(By Taylor's Expansion)

NOTES

Putting values of a and h, we get

$$\sin x = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right)$$

$$+ \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} f'''\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} f^{iv}\left(\frac{\pi}{2}\right) + \ldots\ldots \quad \ldots(1)$$

We have $f(x) = \sin x \quad \therefore \quad f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x \quad \therefore \quad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \quad \therefore \quad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \quad \therefore \quad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{iv}(x) = \sin x \quad \therefore \quad f^{iv}\left(\frac{\pi}{2}\right) = 1$

Putting these values of $f\left(\frac{\pi}{2}\right), f'\left(\frac{\pi}{2}\right), f''\left(\frac{\pi}{2}\right), \ldots\ldots.$ in (1), we get

$$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \ldots\ldots$$

Example 2. *Obtain the first four terms in the expansion of log sin x in powers of (x – 3).*

Sol. Let $f(x) = \log \sin x$

We know that

$f(x) = f(3 + x - 3)$ **(Note this step)**

$= f(a + h)$ where $a = 3$ and $h = x - 3$

$= f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots$

(By Taylor's Expansion)

Putting values of a and h, we have

$$f(x) = f(3) + (x - 3) f'(3) + \frac{(x-3)^2}{2!} f''(3) + \frac{(x-3)^3}{3!} f'''(3) + \ldots\ldots \quad \ldots(1)$$

But $f(x) = \log \sin x$; Putting $x = 3$; $f(3) = \log \sin 3$

$\therefore \quad f'(x) = \frac{1}{\sin x} (\cos x) = \cot x$; $\quad f'(3) = \cot 3$

$f''(x) = -\text{cosec}^2 x$; $f''(3) = -\text{cosec}^2 3$

$f'''(x) = -[2 \cot x (-\text{cosec}^2 x)]$;

$= 2 \cot x \,\text{cosec}^2 x \quad f'''(3) = 2 \cot 3 \,\text{cosec}^2 3$

Putting these values in (1), we get

$$\log \sin x = \log \sin 3 + (x - 3) \cot 3 - \frac{(x-3)^2}{2} \text{cosec}^2 3 + \frac{(x-3)^3}{3} \cot 3 \,\text{cosec}^2 3 + \ldots\ldots$$

NOTES

EXERCISE 3.3

1. (*i*) Expand e^x in power of $(x - 2)$. (*ii*) Expand log x in powers of $x - k$.

(*iii*) Expand a^x in powers of $x - a$. (*iv*) Expand $4x^2 + 7x + 5$ in powers of $x - 3$.

2. Expand log sin x in powers of $(x - 2)$.

3. Use Taylor's Theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x - 2)$.

4. Expand tan x in powers of $\left(x - \frac{\pi}{4}\right)$ upto first four terms.

5. Show that $f(ax) = f(x) + (a - 1)\, x f'(x) + \frac{(a-1)^2 x^2}{2!} f''(x) + \ldots\ldots$

[**Hint.** $f(ax) = f(x + ax - x) = f(x + (a - 1)x] = f(a + h)$ where $a = x$, $h = (a - 1)x$.]

6. If $0 < x \leq 2$, then prove that

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots\ldots$$

Answers

1. (*i*) $e^2\left[1 + (x-2) + \frac{(x-2)^2}{2!} + \ldots\ldots\right]$

(*ii*) $\log k + \frac{1}{k}(x-k) - \frac{1}{2k^2}(x-k)^2 + \frac{1}{3k^3}(x-k)^3 - \ldots\ldots$

(*iii*) $a^a\left[1 + (x-a)\log a + \frac{(x-a)^2}{2!}(\log a)^2 + \frac{(x-a)^3}{3!}(\log a)^3 + \ldots\ldots\right]$

(*iv*) $62 + 31\,(x - 3) + 4\,(x - 3)^2$.

2. $\log \sin 2 + (x-2)\cot 2 - \frac{(x-2)^2}{2}\operatorname{cosec}^2 2 + \frac{(x-2)^3}{3}\cot 2 \operatorname{cosec}^2 2 + \ldots\ldots$

3. $40 + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3$.

4. $1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots$

METHOD OF DIFFERENTIAL EQUATION

Here below we shall explain a method to expand functions like $(\sin^{-1}x)^2$, $e^{a \sin^{-1} x}$, $\sin(m \sin^{-1} x)$, $\cos(m \sin^{-1} x)$ etc.

These functions can also be expanded by the working rule given in Art. 5, but the finding of successive derivatives for these functions is very complicated and hence avoided.

The working rule for the above mentioned functions is being given below :

1. Put the given function equal to y.

2. Find $y_1 = \frac{dy}{dx}$

NOTES

Then (*i*) Take L.C.M. (if possible)

(*ii*) Square both sides if square roots are there.

(*iii*) Try to get y in R.H.S. (if possible).

3. Again differentiate both sides w.r.t. x to get an equation in y_2, y_1, y.

4. Differentiate both sides n times w.r.t. x by Leibnitz Theorem.

[Leibnitz Theorem is

$(uv)_n = {}^nC_0\, u_n \,.\, v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2 \,.\, u_{n-2}\, v_2 + \ldots\ldots + {}^nC_n\, uv_n$.]

5. Put $x = 0$ in equations of steps 1, 2, 3, 4.

6. Put $n = 1, 2, 3, 4$ in last equation of step 5.

7. Now to find $y_n(0)$, discuss the two cases when n is even and when n is odd.

Example 1. *Prove that*

$$\sin(m \sin^{-1} x) = mx + \frac{m(1^2 - m^2)}{3!}x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}x^5 + \ldots\ldots$$

$$+ \ldots\ldots + \frac{m(1^2 - m^2)(3^2 - m^2)\ldots\ldots[(2n-3)^2 - m^2]x^{2n-1}}{(2n-1)!} + \ldots..$$

Sol. Let $y = \sin(m \sin^{-1} x)$ ...(1)

Differentiating w.r.t. x, $y_1 = \cos(m \sin^{-1} x)\dfrac{m}{\sqrt{1-x^2}}$ ...(2)

$\therefore \qquad \sqrt{1-x^2}\; y_1 = m\cos(m \sin^{-1} x)$

Squaring both sides $(1 - x^2)y_1^{\,2} = m^2\cos^2(m\sin^{-1}x) = m^2[1 - \sin^2(m\sin^{-1}x)]$

or $\qquad (1-x^2)y_1^{\,2} = m^2(1 - y^2)$ [By (1)]

Again differentiating w.r.t. x,

$(1 - x^2)2y_1y_2 + y_1^{\,2}(-2x) = m^2(-2yy_1)$ or $2y_1y_2(1 - x^2) - 2xy_1^2 + 2m^2yy_1 = 0$

Cancelling $2y_1$; $y_2(1 - x^2) - xy_1 + m^2y = 0$ ...(3)

Differentiating both sides n times ;

$$\frac{d^n}{dx^n}[y_2(1-x^2)] - \frac{d^n}{dx^n}[y_1 \,.\, x] + \frac{d^n}{dx^n}(m^2y) = 0$$

or $\quad {}^nC_0\,(y_2)_n\,(1 - x^2) + {}^nC_1(y_2)_{n-1}(-2x) + {}^nC_2(y_2)_{n-2}(-2)$

$\qquad - [\,{}^nC_0(y_1)_n \,.\, x + {}^nC_1 \,.\, (y_1)_{n-1} \,.\, 1] + m^2y_n = 0$

or $\quad y_{n+2}(1 - x^2) - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$

$$\left(\because\ {}^nC_0 = 1, {}^nC_1 = n, {}^nC_2 = \frac{n(n-1)}{2!}\right)$$

or $\quad (1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$ ...(4)

Putting $x = 0$ in (1), (2), (3) and (4), we get

$y = \sin(0) = 0 = f(0)$

$y_1 = m\cos 0 = m = f'(0)$

$y_2 = 0 = f''(0)$

$y_{n+2} = (n^2 - m^2)y_n$

Putting $n = 1, 2, 3, 4$ in $y_{n+2} = (n^2 - m^2)y_n$; we get

$y_3 = (1^2 - m^2)y_1 = m(1^2 - m^2) = f'''(0)$

$y_4 = (2^2 - m^2)y_2 = 0 = f^{iv}(0)$

$y_5 = (3^2 - m^2)y_3 = m(1^2 - m^2)(3^2 - m^2) = f^{v}(0)$
$y_6 = (4^2 - m^2)y_4 = 0 = f^{vi}(0)$

Generalising

When n is even y_n (at $x = 0$) $= 0$ ($\because$ $y_2 = 0, y_4 = 0, y_6 = 0$)

NOTES

and when n is odd

y_n (at $x = 0$) $= m(1^2 - m^2)(3^2 - m^2)$ ...... $[(n-2)^2 - m^2]$

In particular y_{2n-1} (Replacing n by $(2n - 1)$ in the above equation

$= m(1^2 - m^2)(3^2 - m^2)$ ...... $[(2n-3)^2 - m^2]$

We know by Maclaurin's Theorem that

$$f(x) = f(0) + xf(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0)$$
$$+ \ldots\ldots + \frac{x^{2n-1}}{(2n-1)!} f^{2n-1}(0) + \ldots\ldots$$

Putting values

$$\sin(m \sin^{-1} x) = mx + \frac{m(1^2 - m^2)}{3!} x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} x^5 + \ldots$$
$$+ \frac{m(1^2 - m^2)(3^2 - m^2) \ldots\ldots [(2n-3)^2 - m^2]}{(2n-1)!} x^{2n-1} + \ldots\ldots$$

Example 2. *If* $y = (\sin^{-1} x)^2$, *show that*

(*a*) $(1 - x^2) y_2 - xy_1 = 2$

(*b*) $\dfrac{(\sin^{-1} x)^2}{2} = \dfrac{x^2}{2!} + \dfrac{2^2}{4!} . x^4 + \dfrac{2^2 . 4^2}{6!} x^6 + \ldots\ldots$

Sol. Let $y = (\sin^{-1} x)^2$ ...(1)

$\therefore$ $y_1 = 2 \sin^{-1} x \times \dfrac{1}{\sqrt{1 - x^2}}$ ...(2)

i.e., $\sqrt{1-x^2} . y_1 = 2\sqrt{y}$. Squaring both sides $(1 - x^2)y_1^2 = 4y$.

Differentiating again,

$$(1 - x^2) . 2y_1y_2 + y_1^2 (-2x) = 4y_1$$

Dividing by $2y_1$, $(1 - x^2)y_2 - xy_1 = 2$ ...(3)

which proves part (*a*).

Differentiating (3), n times by Leibnitz's Theorem,

$$(1 - x^2)y_{n+2} + {}^nC_1 y_{n+1} (-2x) + {}^nC_2 y_n (-2) - [y_{n+1} . x + {}^nC_1 y_n . 1] = 0$$

or $(1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$

or $(1 - x^2)y_{n+2} - (2n + 1) xy_{n+1} = n^2 y_n$ ...(4)

Putting $x = 0$, in (1), (2), (3), (4), we get

$y = 0$, $y_1 = 0$, $y_2 = 2$

$(y_{n+2})_0 = n^2 (y_n)_0$

Putting $n = 1, 2, 3, 4$ in the last equation, we get

$y_3 = 1^2 y_1 = 0$

$y_4 = 2^2 y_2 = 2^2 . 2 = 2.2^2$

$y_5 = 3^2 y_3 = 0$

$y_6 = 4^2 y_4 = 2.2^2 . 4^2$

....

NOTES

$\therefore$ By Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

or
$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \frac{x^6}{6!}(y_6)_0 + \ldots$$

$\therefore$
$$(\sin^{-1} x)^2 = 0 + x(0) + \frac{x^2}{2!} \cdot 2 + \frac{x^3}{3!} \cdot (0) + \frac{x^4}{4!}(2.2^2) + \frac{x^5}{5!}(0)$$
$$+ \frac{x^6}{6!}(2.2^2 \cdot 4^2) + \ldots\ldots$$

or
$$\frac{(\sin^{-1} x)^2}{2} = \frac{x^2}{2!} + \frac{2^2}{4!} x^4 + \frac{2^2 \cdot 4^2}{6!} x^6 + \ldots\ldots$$ which proves part (*b*).

EXERCISE 3.4

1. Show that

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(a^2+1)}{3!} x^3 + \frac{a^2(a^2+2^2)}{4!} x^4 + \frac{a(a^2+1)(a^2+3^2)}{5!} x^5 + \ldots\ldots$$

Hence deduce that $e^\theta = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2}{3!}\sin^3\theta + \ldots\ldots$

[**Hint.** Put $a = 1$ and $\sin^{-1} x = \theta$ *i.e.,* $x = \sin\theta$ in the above expansion]

2. If $y = \cos(m \sin^{-1} x)$, prove that $(y_{n+2})_0 = (n^2 - m^2)(y_n)_0$.

Hence evaluate $(y_n)_0$ and expand $y = \cos m (\sin^{-1} x)$.

3. Prove that $(x + \sqrt{1+x^2})^n = 1 + nx + \frac{n^2x^2}{2!} + n\frac{(n^2-1^2)}{3!}x^3 + \frac{n^2(n^2-2^2)}{4!}x^4 + \ldots\ldots$

Answer

2. If n is odd ; $y_n = 0$

If n is even, $y_n = -m^2(2^2 - m^2)(4^2 - m^2) \ldots\ldots [(n-2)^2 - m^2]$

$$\cos(m \sin^{-1} x) = 1 - \frac{m^2}{2!}x^2 - \frac{m^2(2^2 - m^2)}{4!}x^4 - \frac{m^2(2^2-m^2)(4^2-m^2)}{6!}x^6 - \ldots\ldots$$

UNIT III

4. REDUCTION FORMULAE

NOTES

STRUCTURE

Reduction Formula
Find a reduction formula for $\int \tan^n x\, dx$ and hence find $\int \tan^5 x\, dx$
To find a reduction formula for $\int \cot^n x\, dx$
Find a reduction formula for $\int \sec^n x\, dx$ and hence evaluate
To find a reduction formula for $\int \text{cosec}^n x\, dx$
Find a reduction formula for $\int x^m (\log x)^n\, dx$ and hence evaluate $\int x^m (\log x)^3\, dx$
To find a reduction formula for $\int x^n e^{ax}\, dx$
To find a reduction formula for $\int x^m\, sin\, n^x\, dx$
To find a reduction formula for $\int e^{ax} \sin^n bx\, dx$
To find a reduction formula for $\int \cos^m x \sin nx\, dx$
Find a reduction formula for $\int \sin n\theta/\cos \theta\, d\theta$
To find a reduction formula for $\int \sin^n x\, dx$ and hence evaluate $\int \sin^6 x\, dx$
To find a reduction formula for $\int \cos^n x\, dx$
To evaluate $\int_0^{\pi/2} \sin^n x\, dx$, where n is a positive integer greater than 1
To evaluate $\int_0^{\pi/2} \cos^n x\, dx$, where n is a positive integer greater than 1
Reduction Formulae for $\int \sin^p x \cos^q x\, dx$
Rule for connecting $\int \sin^p x \cos^q x\, dx$ with any one of the above six integrals
Connect $\int \sin^p x \cos^q x\, dx$ with $\int \sin^{p-2} x \cos^q x\, dx$
To evaluate $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$, where p and q are both positive integers greater than 1
Reduction formula for $\int x^m (a + bx^n)^p\, dx$

REDUCTION FORMULA

In Integral Calculus, reduction formula is a formula which connects an integral with another integral which is of the same type but of lower degree or lower order and can be easily integrated. The reduction formulae are generally obtained by applying the rule of integration by the parts. The method is known as integration by *successive reduction*.

NOTES

FIND A REDUCTION FORMULA FOR $\int \tan^n x\, dx$ AND HENCE FIND $\int \tan^5 x\, dx$

$$\int \tan^n x\, dx = \int \tan^{n-2} x \,.\, \tan^2 x\, dx$$
$$= \int \tan^{n-2} x\, (\sec^2 x - 1)\, dx$$
$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} dx$$
$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx \qquad \left(\because \int (f(x))^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1}\right)$$

which is the required reduction formula.

To *evaluate* $\int \tan^5 x\, dx$.

The above reduction formula is $\int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$ ...(1)

Putting $n = 5$ in (1), $\int \tan^5 x\, dx = \dfrac{\tan^4 x}{4} - \int \tan^3 x\, dx$ ...(2)

Putting $n = 3$ in (1), $\int \tan^3 x\, dx = \dfrac{\tan^2 x}{2} - \int \tan x\, dx = \dfrac{\tan^2 x}{2} - \log \sec x$

Putting this value of $\int \tan^3 x\, dx$ in (2),

$$\int \tan^5 x\, dx = \frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \log \sec x\right) = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x.$$

TO FIND A REDUCTION FORMULA FOR $\int \cot^n x\, dx$

$$\int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$$
$$= \int \cot^{n-2} x\, (\text{cosec}^2 x - 1)\, dx = \int \cot^{n-2} x\, \text{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$$
$$= -\int \cot^{n-2} x\, (-\,\text{cosec}^2 x)\, dx - \int \cot^{n-2} x\, dx$$
$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx \qquad \left|\ \because \int (f(x))^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1}\right.$$

which is the required reduction formula.

FIND A REDUCTION FORMULA FOR $\int \sec^n x\, dx$ AND HENCE EVALUATE

$\int \sec^5 x\, dx$.

$$\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx \qquad \text{(Integrate by parts)}$$
$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \,.\, \sec x \tan x \,.\, \tan x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \,.\, \tan^2 x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x\, (\sec^2 x - 1)\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx.$$

Transposing,

$$(1 + n - 2) \int \sec^n x\, dx = \sec^{n-2} x \tan x + (n - 2) \int \sec^{n-2} x\, dx$$

$$\therefore \qquad \int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx \qquad ...(1)$$

NOTES

which is the required reduction formula.

Alternative Form of Art. 4

Use integration by parts to derive the reduction formula :

$$\int \sec^m x\, dx = \frac{1}{m-1} \sec^{m-2} x \tan x + \frac{m-2}{m-1} \int \sec^{m-2} x\, dx.$$

To Evaluate $\int \sec^5 x\, dx$.

Sol. Putting $n = 5$ in (1), we have

$$\int \sec^5 x\, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \int \sec^3 x\, dx \qquad ...(2)$$

Putting $n = 3$ in (1),

$$\int \sec^3 x\, dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x\, dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \log (\sec x + \tan x)$$

$$\therefore \text{ From (2), } \int \sec^5 x\, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \tan x + \frac{3}{8} \log (\sec x + \tan x).$$

TO FIND A REDUCTION FORMULA FOR $\int \text{cosec}^n x\, dx$

$\int \text{cosec}^n x\, dx = \int \text{cosec}^{n-2} x \,.\, \text{cosec}^2 x\, dx$ (Integrate by parts)

$$= \text{cosec}^{n-2} x (-\cot x) - \int (n-2) \text{cosec}^{n-3} x (-\text{cosec } x \cot x)(-\cot x)\, dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) \int \text{cosec}^{n-2} x \cot^2 x\, dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) \int \text{cosec}^{n-2} x (\text{cosec}^2 x - 1)\, dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) \int \text{cosec}^n x\, dx + (n-2) \int \text{cosec}^{n-2} x\, dx$$

Transposing,

$$(1 + n - 2) \int \text{cosec}^n x\, dx = -\text{cosec}^{n-2} x \cot x + (n-2) \int \text{cosec}^{n-2} x\, dx$$

$$\therefore \qquad \int \text{cosec}^n x\, dx = -\frac{\text{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \text{cosec}^{n-2} x\, dx$$

which of the required reduction formula.

Alternative form of Art. 5

Use integration by parts to derive the reduction formula :

$$\int \text{cosec}^m x\, dx = \frac{-1}{m-1} \text{cosec}^{m-2} x \cot x + \frac{m-2}{m-1} \int \text{cosec}^{m-2} x\, dx$$

NOTES

FIND A REDUCTION FORMULA FOR $\int x^m (\log x)^n\, dx$ AND HENCE EVALUATE $\int x^m (\log x)^3\, dx$

$$\int x^m (\log x)^n\, dx = \int (\log x)^n \,.\, x^m\, dx \qquad \text{(Integrate by parts)}$$

$$= (\log x)^n \,.\, \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \,.\, \frac{1}{x} \,.\, \frac{x^{m+1}}{m+1}\, dx$$

$$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1}\, dx \qquad \ldots(1)$$

which is the required reduction formula.

To Evaluate $\int x^m (\log x)^3\, dx$.

Sol. Putting $n = 3$ in (1),

$$\int x^m (\log x)^3\, dx = \frac{x^{m+1}}{m+1} (\log x)^3 - \frac{3}{m+1} \int x^m (\log x)^2\, dx \quad \ldots(2)$$

Putting $n = 2$ in (1),

$$\int x^m (\log x)^2\, dx = \frac{x^{m+1}}{m+1} (\log x)^2 - \frac{2}{m+1} \int x^m \log x\, dx \qquad \ldots(3)$$

Putting $n = 1$ in (1),

$$\int x^m \log x\, dx = \frac{x^{m+1}}{m+1} \log x - \frac{1}{m+1} \int x^m (\log x)^0\, dx$$

$$= \frac{x^{m+1}}{m+1} \log x - \frac{1}{m+1} \int x^m\, dx = \frac{x^{m+1}}{m+1} \log x - \frac{x^{m+1}}{(m+1)^2}.$$

Substituting this value in (3), we have

$$\int x^m (\log x)^2\, dx = \frac{x^{m+1}}{m+1} (\log x)^2 - \frac{2}{m+1} \left[\frac{x^{m+1}}{m+1} \log x - \frac{x^{m+1}}{(m+1)^2}\right]$$

$$= \frac{x^{m+1}}{m+1} (\log x)^2 - \frac{2}{(m+1)^2} x^{m+1} \log x + \frac{2}{(m+1)^3} x^{m+1}.$$

Substituting this value in (2),

$$\int x^m (\log x)^3\, dx = \frac{x^{m+1}}{m+1} (\log x)^3$$

$$- \frac{3}{m+1} \left(\frac{x^{m+1}}{m+1} (\log x)^2 - \frac{2}{(m+1)^2} x^{m+1} \log x + \frac{2}{(m+1)^3} x^{m+1}\right)$$

$$= \frac{x^{m+1}}{m+1} \left((\log x)^3 - \frac{3}{m+1} (\log x)^2 + \frac{6}{(m+1)^2} \log x - \frac{6}{(m+1)^3}\right).$$

Example 1. *If* $\mu_n = \int (\log x)^n\, dx$, *prove that* $\mu_n + n\, \mu_{n-1} = x (\log x)^n$.

Sol. $\mu_n = \int (\log x)^n\, dx = \int (\log x)^n \,.\, 1\, dx$ (Integrate by parts)

$$= (\log x)^n x - \int n (\log x)^{n-1} \,.\, \frac{1}{x}\, x\, dx$$

$$= x (\log x)^n - n \int (\log x)^{n-1}\, dx = x (\log x)^n - n\mu_{n-1}$$

$\therefore \quad \mu_n + n\mu_{n-1} = x (\log x)^n.$

NOTES

TO FIND A REDUCTION FORMULA FOR $\int x^n e^{ax}\, dx$

Integrating by parts, we have

$$\int x^n e^{ax}\, dx = x^n \frac{e^{ax}}{a} - \int nx^{n-1} \cdot \frac{e^{ax}}{a}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx$$

which is the required reduction formula.

Example 2. *Evaluate* $\int_0^\infty x^n e^{-x}\, dx$, *where n is a positive integer.*

Sol. Step I. Let $\quad I_n = \int_0^\infty \underset{\text{I}}{x^n}\, \underset{\text{II}}{e^{-x}}\, dx \qquad \ldots(1)$

Integrating by product Rule $= \left[\frac{x^n e^{-x}}{-1}\right]_0^\infty - \int_0^\infty nx^{n-1} \frac{e^{-x}}{-1}\, dx$

$$= n \int_0^\infty x^{n-1} e^{-x}\, dx \qquad \left[\because \underset{x\to\infty}{\text{Lt}}\, x^n e^{-x} = 0\right]$$

or $\qquad \mathbf{I_n = n\, I_{n-1}} \qquad \ldots(2)$

Step II. Changing n to $(n-1)$ in (2)*, $I_{n-1} = (n-1)\, I_{n-2} \qquad \ldots(3)$

Step III. Putting the value of I_{n-1} from (3) in (2), $I_n = n(n-1)\, I_{n-2}$.

Step IV. Generalising $I_n = [n(n-1)(n-2) \ldots\ldots.\ n \text{ factors}]\, I_{n-n} = n!\, I_0 \qquad \ldots(4)$

Putting $n = 0$ in (1), $\quad I_0 = \int_0^\infty x^0 e^{-x}\, dx = \int_0^\infty e^{-x}\, dx$

or $\qquad I_0 = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -[e^{-\infty} - e^0] = -(0-1) = 1 \qquad (\because \ e^{-\infty} = 0)$

Putting $I_0 = 1$ in (4), $\quad I_n = n!$

EXERCISE 4.1

1. (*a*) If $U_n = \int_0^{\pi/4} \tan^n x\, dx$, show that $U_n + U_{n-2} = \frac{1}{n-1}$ and deduce the value of U_5.

(*b*) If $I_n = \int_0^{\pi/4} \tan^n \theta\, d\theta$, prove that $n(I_{n-1} + I_{n+1}) = 1$ where n is a positive integer.

2. If $I_m = \int_0^1 x^n (\log x)^m\, dx$ ($m \geq 0$ and n being a positive integer), show that $I_m = -\frac{m}{n+1} I_{m-1}$ and hence evaluate I_3.

*This step is being done on comparing the L.H.S. and R.H.S. of equation (2).

NOTES

3. Find a reduction formula for $\int \frac{x^m}{(\log x)^n}\, dx.$

$$\left[\textbf{Hint. } \frac{x^m}{(\log x)^n}\, dx = \int x^{m+1}\left(\frac{(\log x)^{-n}}{x}\right) dx\right].$$

Answers

1. (*a*) $\frac{1}{2}\log 2 - \frac{1}{4}$ **2.** $I_3 = \frac{-6}{(n+1)^4}$

3. $\frac{-x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m}{(\log x)^{n-1}}\, dx.$

TO FIND A REDUCTION FORMULA FOR $\int x^m \sin nx\, dx$

Integrating by parts,

i.e., using the formula $\int \text{I}\,.\,\text{II}\, dx = \text{I}\int \text{II}\, dx - \int \frac{d}{dx}(\text{I})\int \text{II}\, dx\, dx$

$$\int x^m \sin nx\, dx = x^m \,.\, \frac{-\cos nx}{n} - \int mx^{m-1} \,.\, \frac{-\cos nx}{n}\, dx$$

$$= -\frac{x^m \cos nx}{n} + \frac{m}{n}\int x^{m-1}\cos nx\, dx$$

(Integrating again by parts)

$$= -\frac{x^m \cos nx}{n} + \frac{m}{n}\left(x^{m-1}\frac{\sin nx}{n} - \int (m-1)\, x^{m-2}\frac{\sin nx}{n}\, dx\right)$$

$$= -\frac{x^m \cos nx}{n} + \frac{mx^{m-1}\sin nx}{n^2} - \frac{m(m-1)}{n^2}\int x^{m-2}\sin nx\, dx$$

which is the required reduction formula.

TO FIND A REDUCTION FORMULA FOR $\int e^{ax} \sin^n bx\, dx$

$\int e^{ax}\sin^n bx\, dx = \int \underset{\text{I}}{\sin^n bx}\ \underset{\text{II}}{e^{ax}}\, dx$ (Integrate by parts)

$$= \sin^n bx \,.\, \frac{e^{ax}}{a} - \int (n\sin^{n-1} bx \cos bx\,.\,b)\frac{e^{ax}}{a}\, dx$$

$$= \frac{e^{ax}\sin^n bx}{a} - \frac{nb}{a}\int (\sin^{n-1} bx\cos bx)\, e^{ax}\, dx$$ (Integrate again by parts)

$$= \frac{e^{ax}\sin^n bx}{a} - \frac{nb}{a}\left[(\sin^{n-1} bx\cos bx)\frac{e^{ax}}{a}\right.$$

$$\left. - \int \left\{(n-1)\sin^{n-2} bx\cos bx\,.\,b\,(\cos bx) + \sin^{n-1} bx\,(-\sin bx\,.\,b)\right\}\frac{e^{ax}}{a}\, dx\right]$$

NOTES

$$= \frac{e^{ax}\sin^n bx}{a} - \frac{nb}{a}\left[\frac{e^{ax}\sin^{n-1} bx\cos bx}{a} - \int \left\{(n-1)\, b\sin^{n-2} bx\cos^2 bx - b\sin^n bx\right\}\frac{e^{ax}}{a}\,dx\right]$$

$$= \frac{e^{ax}\sin^n bx}{a} - \frac{nb}{a}\left[\frac{e^{ax}\sin^{n-1} bx\cos bx}{a} - \int \left\{(n-1)b\sin^{n-2} bx\,(1-\sin^2 bx) - b\sin^n bx\right\}\frac{e^{ax}}{a}\,dx\right]$$

$$= \frac{e^{ax}\sin^n bx}{a} - \frac{nb}{a}\left[\frac{e^{ax}\sin^{n-1} bx\cos bx}{a} - \int \left((n-1)b\sin^{n-2} bx - nb\sin^n bx\right)\frac{e^{ax}}{a}\,dx\right]$$

$$= \frac{e^{ax}\sin^n bx}{a} - \frac{nb}{a^2}e^{ax}\sin^{n-1} bx\cos bx + \frac{n(n-1)b^2}{a^2}\int e^{ax}\sin^{n-2} bx\,dx - \frac{n^2b^2}{a^2}\int e^{ax}\sin^n bx\,dx$$

Transposing, $\left(1+\frac{n^2b^2}{a^2}\right)\int e^{ax}\sin^n bx\,dx = \frac{ae^{ax}\sin^n bx - nb\,e^{ax}\sin^{n-1} bx\cos bx}{a^2} + \frac{n(n-1)b^2}{a^2}\int e^{ax}\sin^{n-2} bx\,dx.$

$\therefore \quad \int e^{ax}\sin^n bx\,dx$

$$= \frac{e^{ax}\sin^{n-1} bx\,[a\sin bx - nb\cos bx]}{a^2+n^2b^2} + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2} bx\,dx.$$

TO FIND A REDUCTION FORMULA FOR $\int \cos^m x \sin nx\, dx$

Integrating by parts, (Taking sin nx as second function)

$$\int \cos^m x\sin nx\,dx = \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1} x\,(-\sin x)\left(-\frac{\cos nx}{n}\right)dx$$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x\,(\sin x\cos nx)\,dx \qquad ...(1)$$

Now $\sin (n-1)\, x = \sin nx \cos x - \cos nx \sin x$ **(Note this step)**

or $\quad \sin x\cos nx = \sin nx\cos x - \sin (n-1)\, x.$

Substituting this value of $\sin x \cos nx$ in (1), $\int \cos^m x \sin nx\, dx$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x\,[\sin nx\cos x - \sin (n-1)\,x]\,dx$$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^m x\sin nx\,dx + \frac{m}{n}\int \cos^{m-1} x\sin (n-1)\,x\,dx.$$

NOTES

Transposing, $\left(1+\dfrac{m}{n}\right)\displaystyle\int \cos^m x \sin nx\, dx$

$$= -\frac{\cos^m x \cos nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)\, x\, dx$$

$$\therefore \quad \int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \sin (n-1)\, x\, dx.$$

FIND A REDUCTION FORMULA FOR $\int$ sin nθ/cos θ dθ

Let us connect $\displaystyle\int \frac{\sin n\theta}{\cos\theta}\, d\theta$ with $\displaystyle\int \frac{\sin (n-2)\theta}{\cos\theta}\, d\theta$.

Consider $\dfrac{\sin n\theta}{\cos\theta} + \dfrac{\sin (n-2)\theta}{\cos\theta} = \dfrac{\sin n\theta + \sin (n-2)\,\theta}{\cos\theta}$

$$= \frac{2\sin (n-1)\,\theta \cos\theta}{\cos\theta} = 2\sin (n-1)\,\theta$$

$$\left(\because \ \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right)$$

Integrating both sides w.r.t. θ,

$$\int \frac{\sin n\theta}{\cos\theta}\, d\theta + \int \frac{\sin (n-2)\theta}{\cos\theta}\, d\theta = 2\int \sin (n-1)\,\theta\, d\theta = -\frac{2\cos (n-1)\theta}{n-1}$$

$$\therefore \quad \int \frac{\sin n\theta}{\cos\theta}\, d\theta = -\frac{2\cos (n-1)\theta}{n-1} - \int \frac{\sin (n-2)\theta}{\cos\theta}\, d\theta.$$

EXERCISE 4.2

1. Find a reduction formula for $\int x^m \cos nx\, dx$.

2. If $u_n = \displaystyle\int_0^{\pi/2} x^n \sin x\, dx$ and $n > 1$, prove that

   $u_n + n(n-1)\, u_{n-2} = n\left(\dfrac{\pi}{2}\right)^{n-1}$. Hence evaluate u_5.

3. Find a reduction formula for $\int e^{ax} \cos^n x\, dx$.

4. If $I_{m,\,n} = \displaystyle\int_0^{\pi/2} \cos^m x \sin nx\, dx$; show that

   $$I_{m,\,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}.$$

5. (*a*) If $I_{m,\,n} = \displaystyle\int \cos^m x \cos nx\, dx$, prove that

   $$(m+n)\, I_{m,\,n} = \cos^m x \sin nx + m\, I_{m-1,\,n-1}.$$

   (*b*) If $I_{m,\,n} = \displaystyle\int_0^{\pi/2} \cos^m x \cos nx\, dx$, show that $I_{m,\,n} = \dfrac{m}{m+n} I_{m-1,\,n-1}$.

6. (*a*) If $u_n = \int \cos n\theta \operatorname{cosec}\theta\, d\theta$, prove that $u_n - u_{n-2} = \dfrac{2\cos (n-1)\,\theta}{n-1}$.

   (*b*) If $I_n = \int \sin n\theta \sec\theta\, d\theta$, show that $I_n = -\dfrac{2\cos (n-1)\theta}{n-1} - I_{n-2}$.

   [**Hint.** It is Art. 11.]

NOTES

7. (*a*) Find a reduction formula for $\int \frac{\sin nx}{\sin x}\,dx$.

(*b*) Prove that $\int_0^{\pi} \frac{\sin n\theta}{\sin \theta}\,d\theta$ is equal to 0 or π according as n is an even or odd positive integer.

8. Prove that $\int (\sin^{-1} x)^n\,dx = x(\sin^{-1} x)^n + n\sqrt{1-x^2}\,(\sin^{-1} x)^{n-1} - n(n-1)\int (\sin^{-1} x)^{n-2}\,dx$

[Integrate (twice) by Product rule taking 1 as second function.]

Answers

1. $\int x^m \cos nx\,dx = \frac{x^m}{n}\sin nx + \frac{m}{n^2}x^{m-1}\cos nx - \frac{m(m-1)}{n^2}\int x^{m-2}\cos nx\,dx.$

2. $\frac{5\pi^4}{16} - 15\pi^2 + 120.$

3. $\int e^{ax}\cos^n x\,dx = \frac{a\cos x + n\sin x}{a^2+n^2}e^{ax}\cos^{n-1} x + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\,dx.$

7. (*a*) $\int \frac{\sin nx}{\sin x}\,dx = \frac{2\sin(n-1)x}{n-1} + \int \frac{\sin(n-2)x}{\sin x}\,dx.$

TO FIND A REDUCTION FORMULA FOR $\int \sin^n x\,dx$ AND HENCE EVALUATE $\int \sin^6 x\,dx$

$\int \sin^n x\,dx = \int \sin^{n-1} x \,.\, \sin x\,dx$ [Integrate by parts]

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x\cos x\,(-\cos x)\,dx$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

Transposing,

$(1+n-1)\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx.$

$$\therefore \quad \int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx \quad ...(1)$$

which is the required reduction formula.

To Evaluate $\int \sin^6 x\,dx$

Putting $n = 6$ in (1), $\int \sin^6 x\,dx = -\frac{\sin^5 x\,.\cos x}{6} + \frac{5}{6}\int \sin^4 x\,dx$ ...(2)

Again, putting $n = 4$ in (1), $\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx$ ...(3)

Again, putting $n = 2$ in (1), $\int \sin^2 x\,dx = -\frac{\sin x\cos x}{2} + \frac{1}{2}\int \sin^0 x\,dx$

$$= -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int 1\,dx = -\frac{1}{2}\sin x\cos x + \frac{x}{2}$$

Substituting this value in (3), $\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x$

Substituting this value in (2),

$$\int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x.$$

NOTES

TO FIND A REDUCTION FORMULA FOR $\int \cos^n x\, dx$

$\int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$ [Integrate by parts]

$= \cos^{n-1} x \,.\, \sin x - \int (n-1) \cos^{n-2} x\, (-\sin x) \sin x\, dx$

$= \cos^{n-1} x \,.\, \sin x + (n-1) \int \cos^{n-2} x \,.\, \sin^2 x\, dx$

$= \cos^{n-1} x \,.\, \sin x + (n-1) \int \cos^{n-2} x\, (1 - \cos^2 x)\, dx$

$= \cos^{n-1} x \,.\, \sin x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx$

Transposing, $(1 + n - 1) \int \cos^n x\, dx = \cos^{n-1} x \,.\, \sin x + (n-1) \int \cos^{n-2} x\, dx$

$$\therefore \quad \int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$$

which is the required reduction formula.

TO EVALUATE $\int_0^{\pi/2} \sin^n x\, dx$, WHERE n IS A POSITIVE INTEGER GREATER THAN 1

Reproducing Art. 12, we have

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \,.\, \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx.$$

$$\therefore \quad \int_0^{\pi/2} \sin^n x\, dx = \left[-\frac{\sin^{n-1} x \,.\, \cos x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$$

$\left[\text{But } \sin^{n-1} x \cos x = 0, \text{ when } x = \frac{\pi}{2} \text{ or } x = 0 \right]$

$$\therefore \quad \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx \qquad \ldots(1)$$

Changing n to $n - 2$*, in (1)

$$\int_0^{\pi/2} \sin^{n-2} x\, dx = \frac{n-3}{n-2} \int_0^{\pi/2} \sin^{n-4} x\, dx.$$

Substituting this value in (1), $\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)(n-3)}{n(n-2)} \int_0^{\pi/2} \sin^{n-4} x\, dx$ ...(2)

Generalizing from (1) and (2), two cases arise.

Case I. *When n is a + ve odd integer,* then

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)(n-3) \ldots\ldots 2}{n(n-2) \ldots\ldots 3} \int_0^{\pi/2} \sin^1 x\, dx$$

$$= \frac{(n-1)(n-3) \ldots\ldots 2}{n(n-2) \ldots\ldots 3} \qquad \left[\because \int_0^{\pi/2} \sin x\, dx = \Big[-\cos x \Big]_0^{\pi/2} = 1 \right]$$

*This step is being done on comparing the L.H.S. and R.H.S. of equation (1).

Case II. *When n is a + ve even integer,* then

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)(n-3)\ldots\ldots 1}{n(n-2)\ldots\ldots 2} \int_0^{\pi/2} \sin^0 x\, dx = \frac{(n-1)(n-3)\ldots\ldots 1}{n(n-2)\ldots\ldots 2} \cdot \frac{\pi}{2}$$

$$\left[\because \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} 1\, dx = \left(x\right)_0^{\pi/2} = \frac{\pi}{2}\right]$$

Note. $\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1) \times \text{go on diminishing by 2}}{n \times \text{go on diminishing by 2}} \times \boxed{\frac{\pi}{2}}$ **only if n is + ve even integer** $\left(\text{otherwise no } \frac{\pi}{2}\right)$. **Above formula is called Walli's Formula.**

TO EVALUATE $\int_0^{\pi/2} \cos^n x\, dx$, WHERE N IS A POSITIVE INTEGER GREATER THAN 1

Reproducing Art. 13, we have

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$$

$$\therefore \quad \int_0^{\pi/2} \cos^n x\, dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$= \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx \qquad \ldots(1)$$

$$\left[\because \cos^{n-1} x \sin x = 0, \text{ when } x = \frac{\pi}{2} \text{ or } 0\right]$$

Changing n to $n-2$, in (1) $\int_0^{\pi/2} \cos^{n-2} x\, dx = \frac{n-3}{n-2} \int_0^{\pi/2} \cos^{n-4} x\, dx.$

Substituting this value in (1), $\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)}{n(n-2)} \int_0^{\pi/2} \cos^{n-4} x\, dx \qquad \ldots(2)$

Generalizing from (1) and (2), two cases arise.

Case I. *When n is a + ve odd integer,* then

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)\ldots\ldots 2}{n(n-2)\ldots\ldots 3} \int_0^{\pi/2} \cos^1 x\, dx$$

$$= \frac{(n-1)(n-3)\ldots\ldots 2}{n(n-2)\ldots\ldots 3} \qquad \left[\because \int_0^{\pi/2} \cos x\, dx = \left(\sin x\right)_0^{\pi/2} = 1\right]$$

Case II. *When n is a + ve even integer,* then

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)\ldots\ldots 1}{n(n-2)\ldots\ldots 2} \int_0^{\pi/2} \cos^0 x\, dx = \frac{(n-1)(n-3)\ldots\ldots 1}{n(n-2)\ldots\ldots 2} \cdot \frac{\pi}{2}$$

$$\left[\because \int_0^{\pi/2} \cos^0 x\, dx = \int_0^{\pi/2} 1\, dx = \left(x\right)_0^{\pi/2} = \frac{\pi}{2}\right]$$

Note. $\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1) \times \text{go on decreasing by 2}}{n \times \text{go on decreasing by 2}} \times \boxed{\frac{\pi}{2}}$ **only if n is a positive even integer. Above formula is also called Walli's formula.**

Example. *Evaluate*

(*i*) $\int_0^{\pi/2} \sin^6 x\, dx$, (*ii*) $\int_0^{\pi/2} \cos^9 x\, dx$.

NOTES

Sol. (*i*) $\int_0^{\pi/2} \sin^6 x\, dx = \frac{5.3.1}{6.4.2}.\frac{\pi}{2} = \frac{5}{32}\pi$.

(*ii*) $\int_0^{\pi/2} \cos^9 x\, dx = \frac{8.6.4.2}{9.7.5.3} = \frac{128}{315}$.

EXERCISE 4.3

1. If $C_n = \int_0^{\pi/2} \cos^n x\, dx$, show that $C_n = \frac{n-1}{n} C_{n-2}$ where n is any positive integer.

Hence evaluate $\int_0^{\pi/2} \cos^n x\, dx$.

[**Hint.** It is Art 15.]

2. If $u_n = \int_0^{\pi/2} \theta \sin^n \theta\, d\theta$ and $n > 1$, prove that $u_n = \frac{n-1}{n} u_{n-2} + \frac{1}{n^2}$.

Deduce that $u_5 = \frac{149}{225}$.

[**Hint.** $\int \theta \sin^n \theta\, d\theta = \int (\theta \sin^{n-1} \theta) \sin \theta\, d\theta$ and now apply product rule, taking $\theta \sin^{n-1} \theta$ as first function and $\sin \theta$ as second function]

3. If $I_n = \int_0^{\pi/2} x \cos^n x\, dx$, prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$.

REDUCTION FORMULAE FOR $\int \sin^p x \cos^q x\, dx$

$\int \sin^p x \cos^q x\, dx$ can be connected with any one of the following six integrals :

(*i*) $\int \sin^{p-2} x \cos^q x\, dx$ (*ii*) $\int \sin^p x \cos^{q-2} x\, dx$

(*iii*) $\int \sin^{p+2} x \cos^q x\, dx$ (*iv*) $\int \sin^p x \cos^{q+2} x\, dx$

(*v*) $\int \sin^{p-2} x \cos^{q+2} x\, dx$ (*vi*) $\int \sin^{p+2} x \cos^{q-2} x\, dx$.

Thus in finding a reduction formula for $\int \sin^p x \cos^q x\, dx$, we may

(*i*) Decrease or increase by 2 the index of $\sin x$, leaving that of $\cos x$ unchanged (*i*), (*iii*).

(*ii*) Decrease or increase by 2 the index of $\cos x$, leaving that of $\sin x$ unchanged (*ii*), (*iv*).

(*iii*) Decrease the index of $\sin x$ by 2 and increase that of $\cos x$ by (2) (*v*).

(*iv*) Increase the index of $\sin x$ by 2 and decrease that of $\cos x$ by 2 (*vi*).

But we cannot increase or decrease by 2 the indices of both $\sin x$ and $\cos x$ in the same formula.

NOTES

RULE FOR CONNECTING $\int \sin^p x \cos^q x\, dx$ WITH ANY ONE OF THE ABOVE SIX INTEGRALS

Step I. Take $P = \sin^{\lambda+1} x \cos^{\mu+1} x$ where λ is smaller of the two indices of $\sin x$ and μ is smaller of the two indices of $\cos x$ in the two integrals which are to be connected.

Step II. Find $\frac{dP}{dx}$ and express it as a linear function of the two integrands whose integrals are being connected.

Step III. Integrate both sides w.r.t. x, transpose and solve for the given integral.

CONNECT $\int \sin^p x \cos^q x\, dx$ WITH $\int \sin^{p-2} x \cos^q x\, dx$

Hence integrate $\int \sin^4 x \cos^2 x\, dx$.

Sol. (*a*) Let $P = \sin^{p-2+1} x \cos^{q+1} x = \sin^{p-1} x \cos^{q+1} x$

Differentiating both sides w.r.t. x

$$\frac{dP}{dx} = (p-1)\sin^{p-2} x \,.\, \cos x \,.\, \cos^{q+1} x + \sin^{p-1} x \times (q+1)\cos^q x\,(-\sin x)$$

$$= (p-1)\sin^{p-2} x \cos^{q+2} x - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x\, \mathbf{\cos^q x \cos^2 x}^* - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x \cos^q x.\,(1-\sin^2 x) - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x \cos^q x - (p-1)\sin^p x \cos^q x - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x \cos^q x - (p+q)\sin^p x \cos^q x$$

Integrating both sides w.r.t. x

$$P = (p-1)\int \sin^{p-2} x \cos^q x\, dx - (p+q)\int \sin^p x \cos^q x\, dx$$

or $\quad (p+q)\int \sin^p x \cos^q x\, dx = -P + (p-1)\int \sin^{p-2} x \cos^q x\, dx$

$$\therefore \quad \int \sin^p x \cos^q x\, dx = -\frac{\sin^{p-1} x \,.\, \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx \quad \ldots(i)$$

which is the required reduction formula.

(*b*) To integrate $\int \sin^4 x \cos^2 x\, dx$, put $p = 4$, $q = 2$ in (*i*)

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x\, dx \quad \ldots(ii)$$

Again, putting $p = 2$, $q = 2$ in (*i*)

$$\int \sin^2 x \cos^2 x\, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{4}\int \sin^0 x \cos^2 x\, dx$$

$$= -\frac{\sin x \cos^3 x}{4} + \frac{1}{4}\int \frac{1+\cos 2x}{2}\, dx$$

*[**Note.** Now we write $\cos^{q+2} x$ as $\cos^q x \cos^2 x$ and replace $\cos^2 x$ by $1-\sin^2 x$, $\because$ $\cos^{q+2} x$ is not required while $\cos^q x$ is required.]

$$= -\frac{\sin x \cos^3 x}{4} + \frac{1}{8}\left(x + \frac{\sin 2x}{2}\right).$$

Putting this value of $\int \sin^2 x \cos^2 x\, dx$ in (*ii*), we get

NOTES

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{4} - \frac{\sin x \cos^3 x}{8} + \frac{1}{16}\left(x + \frac{\sin 2x}{2}\right).$$

Alternative Form of Art. 18. Find the reduction formula for $\int \sin^p x \cos^q x\, dx$ and hence evaluate $\int \sin^4 x \cos^2 x\, dx$.

Example. *Show that*

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx.$$

Sol. Let $P = \sin^{m+1} x \cos^{n-2+1} x = \sin^{m+1} x \cos^{n-1} x$

$$\therefore \quad \frac{dP}{dx} = (m+1)\sin^m x \,.\, \cos x \,.\, \cos^{n-1} x + \sin^{m+1} x\,(n-1)\cos^{n-2} x\,(-\sin x)$$

$$= (m+1)\sin^m x \cos^n x - (n-1)\sin^{m+2} x \cos^{n-2} x$$

$$= (m+1)\sin^m x \cos^n x - (n-1)\sin^m x \,.\, \sin^2 x \cos^{n-2} x$$

$$= (m+1)\sin^m x \cos^n x - (n-1)\sin^m x\,(1-\cos^2 x)\cos^{n-2} x$$

$$= (m+1)\sin^m x \cos^n x - (n-1)\sin^m x \cos^{n-2} x + (n-1)\sin^m x \cos^n x$$

$$= (m+n)\sin^m x \cos^n x - (n-1)\sin^m x \cos^{n-2} x$$

Integrating both sides,

$$P = (m+n)\int \sin^m x \cos^n x\, dx - (n-1)\int \sin^m x \cos^{n-2} x\, dx$$

or $$(m+n)\int \sin^m x \cos^n x\, dx = P + (n-1)\int \sin^m x \cos^{n-2} x\, dx$$

$$\therefore \quad \int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx$$

which is the required reduction formula.

TO EVALUATE $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$, WHERE p AND q ARE BOTH POSITIVE INTEGERS GREATER THAN 1

Reproducing Art. 18, we have

$$\int \sin^p x \cos^q x\, dx = -\frac{\sin^{p-1} x \,.\, \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx$$

$$\therefore \quad \int_0^{\pi/2} \sin^p x \cos^q x\, dx = -\left[\frac{\sin^{p-1} x \cos^{q+1} x}{p+q}\right]_0^{\pi/2} + \frac{p-1}{p+q}\int_0^{\pi/2} \sin^{p-2} x \cos^q x\, dx$$

$$= \frac{p-1}{p+q}\int_0^{\pi/2} \sin^{p-2} x \cos^q x\, dx \qquad ...(1)$$

$$\left[\because \ \sin^{p-1} x \cos^{q+1} x = 0, \text{ when } x = \frac{\pi}{2} \text{ or } 0\right]$$

Changing p to $p - 2$

$$\int_0^{\pi/2} \sin^{p-2} x \cos^q x\, dx = \frac{p-3}{p+q-2}\int_0^{\pi/2} \sin^{p-4} x \cos^q x\, dx$$

Putting this value in (1),

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)}{(p+q)(p+q-2)} \int_0^{\pi/2} \sin^{p-4} x \cos^q x \, dx \qquad ...(2)$$

Generalising from (1) and (2),

Case I. When p is a + ve odd integer.

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\ldots\ldots 2}{(p+q)(p+q-2)\ldots\ldots(q+3)} \int_0^{\pi/2} \sin^1 x \cos^q x \, dx \qquad ...(3)$$

But $\displaystyle\int_0^{\pi/2} \sin x \cos^q x \, dx = -\int_0^{\pi/2} \cos^q x \, (-\sin x) \, dx$

$$= \left[-\frac{\cos^{q+1} x}{q+1}\right]_0^{\pi/2} = -\frac{1}{q+1}\,[0-1] = \frac{1}{q+1}$$

∴ from (3),

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\ldots\ldots 2}{(p+q)(p+q-2)\ldots\ldots(q+3)(q+1)} \qquad ...(4)$$

Now multiplying both numerator and denominator by

$(q-1)\,.\,(q-3)\,\ldots..$, we have

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{[(p-1)(p-3)\ldots\ldots 2]\,[(q-1)(q-3)\ldots\ldots]}{(p+q)(p+q-2)\ldots\ldots(q+3)(q+1)(q-1)(q-3)\ldots\ldots}.$$

Case II. When p is a + ve even integer.

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\ldots\ldots 1}{(p+q)(p+q-2)\ldots\ldots(q+2)} \int_0^{\pi/2} \sin^0 x \cos^q x \, dx$$

$$= \frac{(p-1)(p-3)\ldots\ldots 1}{(p+q)(p+q-2)\ldots\ldots(q+2)} \int_0^{\pi/2} \cos^q x \, dx \qquad ...(5)$$

Sub-case (*i*) When q is a + ve odd integer.

$$I_{0,\,q} = \int_0^{\pi/2} \cos^q x \, dx = \frac{(q-1)(q-3)\ldots\ldots 2}{q(q-2)\ldots\ldots 3}$$

∴ from (5),

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\ldots\ldots 1}{(p+q)(p+q-2)\ldots\ldots(q+2)} \cdot \frac{(q-1)(q-3)\ldots\ldots 2}{q(q-2)\ldots\ldots 3}$$

Sub-case (*ii*) When q is a + ve even integer.

$$\int_0^{\pi/2} \cos^q x \, dx = \frac{(q-1)(q-3)\ldots\ldots 1}{q(q-2)\ldots\ldots 2} \cdot \frac{\pi}{2}$$

∴ from (5),

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\ldots\ldots 1}{(p+q)(p+q-2)\ldots\ldots(q+2)} \cdot \frac{(q-1)(q-3)\ldots\ldots 1}{q(q-2)\ldots\ldots 2} \times \frac{\pi}{2}.$$

Note. $\displaystyle\int_0^{\pi/2} \sin^p x \cos^q x \, dx$

$$= \frac{[(p-1)\times \text{go on diminishing by } 2]\times[(q-1)\times \text{go on diminishing by } 2]}{(p+q)\times \text{go on diminishing by } 2} \times \boxed{\frac{\pi}{2}}$$

only if both p and q are + ve even integers $\left(\textbf{otherwise no } \frac{\pi}{2}\right)$.

Example 1. (*i*) *Evaluate* $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$

(*ii*) $\int_0^{\pi/2} \sin^8 x \cos^4 x\, dx$

NOTES

(*iii*) $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx$

(*iv*) $\int_0^{\pi/2} \cos^7 \theta \sin\theta\, d\theta$.

Sol. (*i*) $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$ [Here $p = 4$, $q = 2$ are both even, $p + q = 6$]

$$= \frac{3 \times 1.1}{6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{\pi}{32}.$$

(*ii*) $\int_0^{\pi/2} \sin^8 x \cos^4 x\, dx = \frac{7.5.3.1 \times 3.1}{12.10.8.6.4.2} \times \frac{\pi}{2} = \frac{7\pi}{2048}$.

(*iii*) $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{4.2.3.1}{9.7.5.3.1}$ (Here $p = 5$, $q = 4$ and $p + q = 5 + 4 = 9$)

$$= \frac{8}{315}.$$

(*iv*) $\int_0^{\pi/2} \cos^7 \theta \sin \theta\, d\theta$

Here $p = 1$.

$\therefore$ Formula of Art. 19 is not applicable.

$$\therefore \int_0^{\pi/2} \cos^7 \theta \sin \theta\, d\theta = -\int_0^{\pi/2} (\cos \theta)^7 (-\sin \theta\, d\theta)$$

$$= -\left[\frac{(\cos \theta)^8}{8}\right]_0^{\pi/2} \qquad \left|\because \int \mathbf{(f(\theta))^n\, f'(\theta)}\, d\theta = \frac{\mathbf{(f(\theta))^{n+1}}}{\mathbf{n+1}}\right.$$

$$= \frac{-1}{8}\left[\left(\cos \frac{\pi}{2}\right)^8 - (\cos 0)^8\right] = \frac{-1}{8}(0 - 1) = \frac{1}{8}.$$

Example 2. *Evaluate* $\int_0^1 x^6 \sin^{-1} x\, dx$.

Sol. Put $\sin^{-1} x = \theta$

$\therefore$ $x = \sin \theta$

$\therefore$ $dx = \cos \theta\, d\theta$

To change the limits

When $x = 0$, $\sin \theta = 0$ $\therefore$ $\theta = 0$

When $x = 1$, $\sin \theta = 1$ $\therefore$ $\theta = \frac{\pi}{2}$

$$\therefore \int_0^1 x^6 \sin^{-1} x\, dx = \int_0^{\pi/2} \sin^6 q\, .\, q\, .\, \cos q\, dq$$

$$= \int_0^{\pi/2} \underset{\text{I}}{\theta}\, .\, \sin^6 \theta \underset{\text{II}}{\cos} \theta\, d\theta \qquad \text{(Product Rule)}$$

$$= \left(\theta \int_0^{\pi/2} \sin^6 \theta \cos \theta\, d\theta - \int_0^{\pi/2} \frac{d}{d\theta}(\theta) \int \sin^6 \theta \cos \theta\, d\theta\, d\theta\right.$$

$$= \left(\theta \frac{\sin^7 \theta}{7}\right)_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin^7 \theta\, d\theta}{7}$$

NOTES

(See Example 1 (*iv*) above)

$$= \frac{\pi}{2} \cdot \frac{1}{7} - \frac{1}{7} \cdot \frac{6.4.2}{7.5.3} = \frac{\pi}{14} - \frac{16}{245}.$$

Example 3. *Prove that* $\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{2n!}{(2^n (n)!)^2} \frac{\pi}{2}$.

Sol. We know by Art. 14 that

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{(2n-1)(2n-3)\ldots\ldots 1}{2n(2n-2)\ldots\ldots 2} \times \frac{\pi}{2}.$$

Multiplying both numerator and denominator by denominator *i.e.*, by $2n(2n-2) \ldots\ldots 2$,

$$= \frac{2n(2n-1)(2n-2)(2n-3)\ldots\ldots 2.1}{[2n(2n-2)\ldots\ldots 2]^2} \cdot \frac{\pi}{2}$$

$$= \frac{2n!}{[2.n.2(n-1)\ldots\ldots 2.1]^2} \frac{\pi}{2}$$

$$= \frac{2n!}{[2^n\, n(n-1)\ldots\ldots 1]^2} \frac{\pi}{2} = \frac{2n!}{(2^n\,(n)!)^2} \frac{\pi}{2}.$$

Example 4. *Prove that*

$$\int_0^{2a} x^n \sqrt{2ax - x^2}\, dx = \pi a^{n+2} \frac{(2n+1)(2n-1)\ldots\ldots 3.1}{(n+2)(n+1)\, n \ldots\ldots 2.1}.$$

where n is a natural number.

Sol. L.H.S. $= \int_0^{2a} x^n \sqrt{2ax - x^2}\, dx = \int_0^{2a} x^n \sqrt{x(2a-x)}\, dx$

$$= \int_0^{2a} x^n \sqrt{x} \sqrt{2a - x}\, dx \qquad \ldots(1)$$

Put **x = 2a sin² θ**

$\therefore \quad dx = 2a \,.\, 2 \sin\theta \cos\theta \, d\theta$

$= 4a \sin\theta \cos\theta \, d\theta$

To change the limits

when $x = 0$, $2a \sin^2\theta = 0$. But $2a \neq 0 \quad \therefore \quad \sin\theta = 0$

$\therefore \quad \theta = 0$

when $x = 2a$, $2a \sin^2\theta = 2a$ or $\sin^2\theta = 1 = \sin^2 \frac{\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{2}$.

Putting these values in (1),

$$\text{L.H.S.} = \int_0^{\pi/2} (2a \sin^2\theta)^n \sqrt{2a \sin^2\theta} \sqrt{2a - 2a \sin^2\theta} \,.\, 4a \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (2a)^n \sin^{2n}\theta \,.\, \sqrt{2a} \sin\theta \,.\, \sqrt{2a(1 - \sin^2\theta)}\; 4a \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (2a)^n \sin^{2n}\theta \sqrt{2a} \sin\theta \sqrt{2a} \cos\theta \,.\, 4a \sin\theta \cos\theta \, d\theta$$

NOTES

$$= (2a)^n \sqrt{2a}\ \sqrt{2a}\ 4a \int_0^{\pi/2} \sin^{2n+2}\theta \cos^2\theta\, d\theta$$

$$= (2a)^{n+1} \cdot 2 \cdot 2a \cdot \frac{(2n+1)(2n-1)\ldots\ldots 3.1.1}{(2n+4)(2n+2)\,2n\ldots\ldots 2} \cdot \frac{\pi}{2}$$

$$= 2^{n+2} \cdot a^{n+2} \frac{(2n+1)(2n-1)\ldots\ldots 3.1}{2(n+2)\,2\,(n+1)\ldots\ldots 2.n\ldots\ldots 2.1}\pi$$

$$= 2^{n+2}\, a^{n+2} \frac{(2n+1)(2n-1)\ldots\ldots 3.1}{2^{n+2}(n+2)(n+1).n.1}\pi$$

$$= \pi a^{n+2} \frac{(2n+1)(2n-1)\ldots\ldots 3.1}{(n+2)(n+1).n\ldots\ldots 1}.$$

EXERCISE 4.4

1. Prove that $\int_0^{\pi/2} \sin^{10} x\, dx = \int_0^{\pi/2} \cos^{10} x\, dx = \frac{63\pi}{512}$.

2. Prove that (*i*) $\int_0^{\pi/2} \sin^5 x \cos^5 x\, dx = \frac{1}{60}$

(*ii*) $\int_0^{\pi/2} \sin^4\theta \cos^8\theta\, d\theta = \frac{7\pi}{2048}$ (*iii*) $\int_0^{\pi/2} \sin^9\theta \cos\theta\, d\theta = \frac{1}{10}$

(*iv*) $\int_0^{\pi/2} \sin^5 x \cos^6 x\, dx = \frac{8}{693}$.

3. Prove that $\int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2}\, dx = \frac{5\pi}{2^{11}}$.

$\left[\textbf{Hint.}\ \text{Put}\ \frac{x}{2} = t.\right]$

4. Prove that (*i*) $\int_0^{a} x^4 \sqrt{a^2 - x^2}\, dx = \frac{\pi a^6}{32}$. (*ii*) $\int_0^{a} \frac{x^4}{\sqrt{a^2 - x^2}}\, dx = \frac{3a^4\pi}{16}$.

[**Hint.** Put $x = a \sin\theta$.]

5. (*i*) Evaluate $\int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$ [**Hint.** put $x = 2a\sin^2\theta$]

(*ii*) Evaluate $\int_0^{2} x^{5/2} \sqrt{2 - x}\, dx$

(*iii*) Prove that $\int_0^{4} x^3 \sqrt{4x - x^2}\, dx = 28\pi$

(*iv*) Prove that $\int_0^{2a} x^{9/2} (2a - x)^{-1/2}\, dx = \frac{63\pi a^5}{8}$.

6. (*i*) Evaluate $\int_0^{\infty} \frac{dx}{(1 + x^2)^n}$.

(*ii*) $\int_0^{1} \frac{x^{2n}}{\sqrt{1 - x^2}}\, dx$ and hence find the sum of the series

$$\frac{1}{2n+1} + \frac{1}{2} \cdot \frac{1}{2n+3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2n+5} + \ldots\ \text{to}\ \infty.$$

$$\left[\ \textbf{Hint.}\ \int_0^1 x^{2n}(1-x^2)^{-1/2}\,dx\right.$$

$$= \int_0^1 x^{2n}\left[1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{3}{4}x^4+\ldots\ldots\infty\right]dx$$

$$= \int_0^1 \left[x^{2n}+\frac{1}{2}x^{2n+2}+\frac{1}{2}\cdot\frac{3}{4}x^{2n+4}+\ldots\ldots\infty\right]dx$$

$$= \left[\frac{x^{2n+1}}{2n+1}+\frac{1}{2}\cdot\frac{x^{2n+3}}{2n+3}+\frac{1}{2}\cdot\frac{3}{4}\,\frac{x^{2n+5}}{2n+5}+\ldots\ldots\infty\right]_0^1$$

$$\left.= \frac{1}{2n+1}+\frac{1}{2}\cdot\frac{1}{2n+3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{2n+5}+\ldots..\ \right].$$

7. Evaluate $\int_0^1 x^5\sqrt{\dfrac{1+x^2}{1-x^2}}\,dx.$

[**Hint.** Put $x^2 = \cos\theta$.]

8. Evaluate $\int_0^1 x^5\sin^{-1}x\,dx.$

9. If $I_{p,\,q} = \int_0^{\pi/2}\sin^p x\cos^q x\,dx$, prove that

$$I_{p,\,q} = \frac{(p-1)}{(p+q)}\,\frac{(p-3)(p-5)\ldots\ldots 2\,.\,1}{(p+q-2)(p+q-4)\ldots\ldots(q+3)(q+1)}$$

where p **is an odd positive integer and** q is a positive integer, even or odd.

[**Hint.** Reproduce Art. 19 upto Eqn. (4).]

Answers

5. (*i*) $\dfrac{5\pi}{8}a^4$ (*ii*) $\dfrac{5\pi}{8}$.

6. (*i*) $\dfrac{(2n-3)(2n-5)\ldots..3\,.\,1}{(2n-2)(2n-4)\ldots.4\,.\,2}\cdot\dfrac{\pi}{2}$ (*ii*) $\dfrac{(2n-1)(2n-3)\ldots..3\,.\,1}{2^n\,.\,n\,!}\cdot\dfrac{\pi}{2}$.

7. $\dfrac{3\pi+8}{24}$ **8.** $\dfrac{11\pi}{192}$.

REDUCTION FORMULA FOR $\int x^m (a + bx^n)^p\, dx$

$\int x^m (a + bx^n)^p\, dx$ can be connected with any one of the following six integrals :

(*i*) $\int x^m (a + bx^n)^{p-1}\, dx$ (*ii*) $\int x^{m-n} (a + bx^n)^p\, dx$

(*iii*) $\int x^m (a + bx^n)^{p+1}\, dx$ (*iv*) $\int x^{m+n} (a + bx^n)^p\, dx$

(*v*) $\int x^{m+n} (a + bx^n)^{p-1}\, dx$ (*vi*) $\int x^{m-n} (a + bx^n)^{p+1}\, dx$.

Thus the index m of the monomial x^m can only be increased or decreased by the index n of x^n in the binomial $a + bx^n$ and the index p of the binomial $a + bx^n$ can be increased or decreased by one only. The indices of binomial $a + bx^n$ and the monomial x^m cannot both be decreased or increased in the same formula.

NOTES

Example 1. *Connect* $\int x^m (a + bx^n)^p\, dx$ *with* $\int x^m (a + bx^n)^{p-1}\, dx$.

Sol. Let $\quad P = x^{m+1} (a + bx^n)^{p-1+1} = x^{m+1}(a + bx^n)^p$

$$\therefore \quad \frac{dP}{dx} = (m + 1)\, x^m (a + bx^n)^p + x^{m+1} \,.\, p\, (a + bx^n)^{p-1} \,.\, bn\, x^{n-1}$$

$$= (m + 1)\, x^m (a + bx^n)^p + pn\, bx^{m+n} (a + bx^n)^{p-1}$$

$$= (m + 1)\, x^m (a + bx^n)^p + pnx^m \,.\, bx^n(a + bx^n)^{p-1}$$

$$= (m + 1)\, x^m (a + bx^n)^p + pnx^m \,.\, (a + \overline{bx^n}\, * - a)\, (a + bx^n)^{p-1}$$

$$= (m + 1)\, x^m (a + bx^n)^p + pnx^m (a + bx^n)^p - pnax^m (a + bx^n)^{p-1}$$

$$= (m + 1 + pn)\, x^m (a + bx^n)^p - pnax^m (a + bx^n)^{p-1}.$$

Integrating both sides,

$$P = (m + 1 + pn) \int x^m (a + bx^n)^p\, dx - pna \int x^m (a + bx^n)^{p-1}\, dx$$

$$\therefore \quad (m + 1 + pn) \int x^m(a + bx^n)^p\, dx = P + pna \int x^m (a + bx^n)^{p-1}\, dx$$

$$\therefore \quad \int x^m (a + bx^n)^p\, dx = \frac{x^{m+1} (a + bx^n)^p}{m + 1 + pn} + \frac{pna}{m + 1 + pn} \times \int x^m (a + bx^n)^{p-1}\, dx$$

which is the required reduction formula.

Example 2. *Prove that*

$$\int \frac{dx}{(a^2 + x^2)^n} = \frac{x}{2a^2(n-1)(a^2 + x^2)^{n-1}} + \frac{(2n-3)}{2a^2\, (n-1)} \int \frac{dx}{(a^2 + x^2)^{n-1}} .$$

Sol. Let us connect $\displaystyle\int \frac{dx}{(a^2 + x^2)^n} = \int x^0\, (a^2 + x^2)^{-n}\, dx$

with $\displaystyle\int \frac{dx}{(a^2 + x^2)^{n-1}} = \int x^0\, (a^2 + x^2)^{-n+1}\, dx.$

Let $\quad P = x^{0+1}\, (a^2 + x^2)^{-n+1}$ | "Smaller Index + 1" Method

or $\quad P = x\, (a^2 + x^2)^{-n+1} = \dfrac{x}{(a^2 + x^2)^{n-1}}$

Diff. both sides w.r.t. x,

$$\frac{dP}{dx} = (a^2 + x^2)^{-n+1} + x\, (-n + 1)\, (a^2 + x^2)^{-n} \,.\, 2x$$

$$= \frac{1}{(a^2 + x^2)^{n-1}} + 2(1 - n)\, \frac{x^2}{(a^2 + x^2)^n}$$

$$= \frac{1}{(a^2 + x^2)^{n-1}} + 2(1 - n)\, \frac{[(a^2 + x^2) - a^2]}{(a^2 + x^2)^n}$$

or $$\frac{dP}{dx} = \frac{1}{(a^2 + x^2)^{n-1}} + \frac{2(1-n)}{(a^2 + x^2)^{n-1}} - \frac{2a^2\,(1-n)}{(a^2 + x^2)^n}$$

$$= \frac{3 - 2n}{(a^2 + x^2)^{n-1}} - \frac{2a^2\,(1-n)}{(a^2 + x^2)^n}$$

*[**Note.** Now x^{m+n} is not required $\therefore$ we write $bx^{m+n} = x^m \,.\, x^n \,.\, b$.

x^n is also not required but powers of $a + bx^n$ are required.

$\therefore$ We write $bx^{m+n} = x^m \,.\, bx^n = x^m\, (a + bx^n - a)$].

Integrating both sides w.r.t. x, we have

$$P = (3 - 2n) \int \frac{1}{(a^2 + x^2)^{n-1}}\, dx - 2a^2 (1 - n) \int \frac{1}{(a^2 + x^2)^n}\, dx$$

Transposing,

$$2a^2 (1 - n) \int \frac{1}{(a^2 + x^2)^n}\, dx = - P + (3 - 2n) \int \frac{1}{(a^2 + x^2)^{n-1}}\, dx$$

Putting the value of P and dividing both sides by $2a^2 (1 - n)$

$$\int \frac{1}{(a^2 + x^2)^n}\, dx = - \frac{x}{2a^2 (1-n)(a^2 + x^2)^{n-1}} - \frac{(2n - 3)}{2a^2(1-n)} \int \frac{1}{(a^2 + x^2)^{n-1}}\, dx$$

$$= \frac{x}{2a^2 (n - 1)(a^2 + x^2)^{n-1}} + \frac{(2n - 3)}{2a^2(n - 1)} \int \frac{1}{(a^2 + x^2)^{n-1}}\, dx.$$

Example 3. *Obtain a reduction formula for* $\int \frac{x^n\, dx}{(x^3 - 1)^{1/3}}$ *and find the value of* $\int x^8 (x^3 - 1)^{-1/3}\, dx$.

Sol. Let us connect the given integral

$$\int x^n (x^3 - 1)^{-1/3}\, dx \text{ with the integral } \int x^{n-3} (x^3 - 1)^{-1/3}\, dx$$

[**Note.** In the second integral we have taken x^{n-3} because of the presence of x^3 in the bracket]

Let $\quad P = x^{n-3+1} (x^3 - 1)^{-\frac{1}{3}+1}$ | "smaller index +1" method

or $\quad P = x^{n-2}(x^3 - 1)^{2/3}$

Diff. both sides w.r.t x, we have

$$\frac{dP}{dx} = x^{n-2} \cdot \frac{2}{3} (x^3 - 1)^{-1/3} \cdot 3x^2 + (n - 2)x^{n-3} (x^3 - 1)^{2/3}$$

or $$\frac{dP}{dx} = 2x^n (x^3 - 1)^{-1/3} + (n - 2)x^{n-3}(x^3 - 1) (x^3 - 1)^{2/3-1} \qquad \textbf{(Note)}$$

or $$\frac{dP}{dx} = 2x^n (x^3 - 1)^{-1/3} + (n - 2) (x^n - x^{n-3}) (x^3 - 1)^{-1/3}$$

or $$\frac{dP}{dx} = 2x^n (x^3 - 1)^{-1/3} + (n - 2)\, x^n (x^3 - 1)^{-1/3} - (n - 2)x^{n-3}(x^3 - 1)^{-1/3}$$

$$= (2 + n - 2)\, x^n (x^3 - 1)^{-1/3} - (n - 2)x^{n-3} (x^3 - 1)^{-1/3}$$

Integrating both sides w.r.t. x, we have

$$P = n\int x^n (x^3 - 1)^{-1/3}\, dx - (n - 2) \int x^{n-3} (x^3 - 1)^{-1/3}\, dx$$

$$\therefore \quad n \int x^n (x^3 - 1)^{-1/3}\, dx = P + (n - 2)\int x^{n-3}(x^3 - 1)^{-1/3}\, dx$$

Putting the value of P and dividing both sides by n,

$$\int x^n (x^3 - 1)^{-1/3}\, dx = \frac{x^{n-2} (x^3 - 1)^{2/3}}{n} + \frac{(n - 2)}{n}\int x^{n-3}(x^3 - 1)^{-1/3}\, dx \qquad \ldots(1)$$

which is the required reduction formula.

NOTES

To evaluate $\int x^8 (x^3 - 1)^{-1/3}\, dx$

Putting $n = 8$ in (1), $\int x^8(x^3-1)^{-1/3}\,dx = \dfrac{x^6(x^3-1)^{2/3}}{8} + \dfrac{6}{8}\int x^5(x^3-1)^{-1/3}\,dx$ ...(2)

Putting $n = 5$ in (1), $\int x^5(x^3-1)^{-1/3}\,dx = \dfrac{x^3(x^3-1)^{2/3}}{5} + \dfrac{3}{5}\int x^2(x^3-1)^{-1/3}\,dx$ ...(3)

Now $\int x^2(x^3-1)^{-1/3}\,dx = \dfrac{1}{3}\int (x^3-1)^{-1/3}(3x^2)\,dx$

$$= \frac{1}{3}\frac{(x^3-1)^{2/3}}{\frac{2}{3}} + c$$

$$\left[\because \int (f(x)\, f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} \text{ if } n \neq -1\right]$$

$$= \frac{1}{2}(x^3-1)^{2/3} + c$$

Putting this value in eqn. (3), we have

$$\int x^5(x^3-1)^{-1/3}\,dx = \frac{x^3(x^3-1)^{2/3}}{5} + \frac{3}{10}(x^3-1)^{2/3} + c'$$

$$\int x^8(x^3-1)^{-1/3}\,dx = \frac{x^6(x^3-1)^{2/3}}{8} + \frac{3}{4}\left[\frac{x^3(x^3-1)^{2/3}}{5} + \frac{3}{10}(x^3-1)^{2/3}\right] + C$$

$$= \frac{1}{8}x^6(x^3-1)^{2/3} + \frac{3}{20}x^3(x^3-1)^{2/3} + \frac{9}{40}(x^3-1)^{2/3} + C$$

$$= (x^3-1)^{2/3}\left[\frac{x^6}{8} + \frac{3x^3}{20} + \frac{9}{40}\right] + C.$$

EXERCISE 4.5

1. Connect $\int x^m (a + bx^n)^p\, dx$ with $\int x^{m+n} (a + bx^n)^p\, dx$.
2. If n is a positive integer, prove that

$$\int (a^2 + x^2)^{n/2}\, dx = \frac{x(a^2+x^2)^{n/2}}{n+1} + \frac{na^2}{n+1}\int (a^2+x^2)^{n/2-1}\,dx.$$

3. If $I_n = \int x^n \sqrt{a^2 - x^2}\, dx$, show that $I_n = \dfrac{-x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \left(\dfrac{n-1}{n+2}\right)a^2\, I_{n-2}$.
4. If $I_n = \int x^n \sqrt{a - x}\, dx$, prove that

   $(2n + 3)\, I_n = 2an\, I_{n-1} - 2x^n (a - x)^{3/2}$ and hence evaluate $\int_0^a x^2 \sqrt{ax - x^2}\, dx$.
5. If I_n denotes $\int_0^a (a^2 - x^2)^n\, dx$ and $n > 0$, prove that $I_n = \dfrac{2na^2}{2n+1}\, I_{n-1}$.

   Hence evaluate $\int_0^a (a^2 - x^2)^3\, dx$.

NOTES

6. If $I_{m,n} = \int_0^1 x^m (1-x)^n \, dx$, prove that $(m + n + 1) I_{m,n} = n I_{m,n-1}$.

Deduce the value of $\int_0^1 x^4 (1-x)^3 \, dx$.

7. If m is a positive integer, find a reduction formula for $\int x^m \sqrt{2ax - x^2} \, dx$.

Hence find the value of $\int_0^{2a} x^3 \sqrt{2ax - x^2} \, dx$.

$\left[\textbf{Hint}. \; x^m \sqrt{2ax - x^2} = x^m \sqrt{x} \sqrt{2a - x} = x^{m+\frac{1}{2}} \sqrt{2a - x}. \right]$

Answers

1. $\int x^m (a + bx^n)^p \, dx = \dfrac{x^{m+1} (a + bx^n)^{p+1}}{(m+1)a} - \dfrac{b(pn + n + m + 1)}{(m+1)a} \int x^{m+n} (a + bx^n)^p \, dx.$

4. $\dfrac{5\pi a^4}{128}$ **5.** $\dfrac{16a^7}{35}$ **6.** $\dfrac{1}{280}$.

7. $\int x^m \sqrt{2ax - x^2} \, dx = -x^{m-1} \dfrac{(2ax - x^2)^{3/2}}{m+2} + a \dfrac{(2m+1)}{m+2} \int x^{m-1} \sqrt{2ax - x^2} \, dx \,; \; \dfrac{7}{8} \pi a^5.$

NOTES

5. RECTIFICATION

STRUCTURE

RECTIFICATION

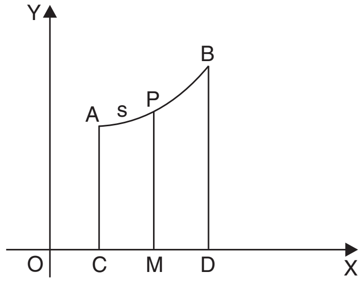
Definition. The process of finding the length of an arc of a curve between two given points on it is called *rectification*.

LENGTH OF THE CURVE WHEN ITS CARTESIAN EQUATION IS GIVEN

Prove that the length of the arc of the curve **y = f(x)**, *between the two points whose abscissae are a and b is given by*

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

where y and *dy/dx* are *continuous* and *single-valued functions in the interval* [*a*, *b*] *and the integrand does not change sign in this interval.*



Let AB be the curve $y = f(x)$ and A, B are the points whose abscissae are a and b respectively and CA, DB are their ordinates.

Let P(x, y) be *any* point on the curve and draw MP $\perp$ x-axis.

If s denotes the length of the arc AP measured from the fixed point A to the variable point P, then s is clearly a function of x.

From Differential Calculus, we know that

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad \ldots(1)$$

$$\therefore \quad \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \frac{ds}{dx}\, dx = \Big[\, s \,\Big]_a^b$$

$$= (\text{value of } s \text{ at } x = b) - (\text{value of } s \text{ at } x = a)$$

$$= \text{arc AB} - 0 = \text{arc AB}.$$

Hence arc AB $= \displaystyle\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$

Remark. It must be noted that in fact, $\dfrac{ds}{dx} = \pm\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

We take the + ve sign with the radical under the assumption that s increases with x. The sign is – ve if s decreases with the increase of x.

If $\dfrac{ds}{dx}$ or the integrand changes sign at some value c within the range of integration $[a, b]$, then the definite integral from a to b must be broken into the sum of two definite integrals, one from a to c and other from c to b and the +ve value of the integrand taken in each. Otherwise, the result will be the difference of the lengths of two arcs.

Cor. 1. From (1), we have $s = \displaystyle\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$

Cor. 2. *If the equation of the curve is of the form* **x = f(y),** *then the length of the arc between the points whose ordinates are c and d is*

$$\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

[**Hint.** Proceed as in Art. 2 and instead of (1) use $\dfrac{ds}{dy} = \sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$.]

Caution. To find the length of an arc of a cartesian curve, we must express either **y as an explicit function of x or x as an explicit function of y.**

Secondly, if we use the formula, $s = \displaystyle\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$, then $\dfrac{dy}{dx}$ must be a function of x alone.

If we use the formula $s = \displaystyle\int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$, then $\dfrac{dx}{dy}$ must be a function of y alone.

METHOD TO FIND THE LENGTH OF AN ARC OF A CARTESIAN CURVE

Step I. To find Limits of Integration. If the end points of the arc (whose length we are to find) are given ; then we know the limits of integration.

In such problems we need not trace the curve. Otherwise, trace the curve roughly to know the limits of integration. (While tracing the curve here, **Symmetry** of the curve and its **points of Intersection with the axes** are the two main points which should be discussed).

Step II. Express y as an explicit function of x or x as an explicit function of y.

Step III. Find $\frac{dy}{dx}$ or $\frac{dx}{dy}$.

Step IV. Use the formula, $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ or $\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$.

Example 1. *Show that in the catenary $y = c \cosh x/c$, the length of the arc from the vertex (0, c) to any point (x, y) is given by (i) $s = c \sinh x/c$ and (ii) $s^2 = y^2 - c^2$.*

Sol. The equation of the curve is $y = c \cosh \frac{x}{c}$

$\therefore \quad \frac{dy}{dx} = c \sinh \frac{x}{c} \cdot \frac{1}{c} = \sinh \frac{x}{c}.$

(*i*) $\therefore$ Reqd. length of arc from $x = 0$ to $x = x$ is

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^x \sqrt{1+\sinh^2 \frac{x}{c}}\, dx$$

$$= \int_0^x \cosh \frac{x}{c}\, dx = c\left[\sinh \frac{x}{c}\right]_0^x = c\left(\sinh \frac{x}{c} - 0\right) = c \sinh \frac{x}{c}.$$

(*ii*) Now $s^2 = c^2 \sinh^2 \frac{x}{c} = c^2\left[\cosh^2 \frac{x}{c} - 1\right] = c^2\left[\left(\frac{y}{c}\right)^2 - 1\right] = y^2 - c^2$

which proves the required result.

Example 2. *Find the length of arc of the parabola $y^2 = 4ax$*

(*i*) *from the vertex to an extremity of the latus rectum.*

(*ii*) *cut off by the latus rectum.*

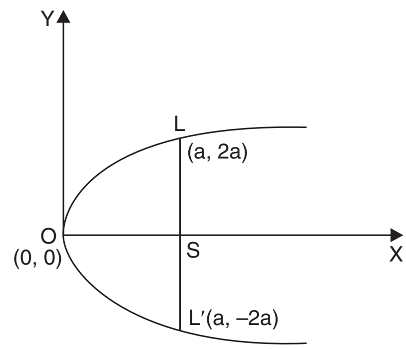
Sol. The equation of the parabola is $y^2 = 4ax$ ...(1)

Differentiating, $2y \frac{dy}{dx} = 4a$ or $\frac{dx}{dy} = \frac{y}{2a}$

Let O be the vertex and L be an extremity of latus rectum (a line through the focus and ⊥ to axis).

Extremities of latus rectum of the parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$.

At O, we have $y = 0$ and at L, we have $y = 2a$.



(*i*) $\therefore$ Length of arc OL

$$= \int_0^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

NOTES

$$= \int_0^{2a} \sqrt{1 + \frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a} \int_0^{2a} \sqrt{y^2 + 4a^2}\, dy$$

$$= \frac{1}{2a} \left[\frac{1}{2} y \sqrt{y^2 + 4a^2} + 2a^2 . \sinh^{-1} \frac{y}{2a} \right]_0^{2a}$$

$$\left(\because \int \sqrt{\mathbf{x^2 + a^2}}\, \mathbf{dx} = \frac{\mathbf{x} \sqrt{\mathbf{x^2 + a^2}}}{\mathbf{2}} + \frac{\mathbf{a^2}}{\mathbf{2}} \, \mathbf{\sinh^{-1} \frac{x}{a}} \right)$$

$$= \frac{1}{2a} \left[a \sqrt{8a^2} + 2a^2 \sinh^{-1} 1 - 0 \right] = a\, [\sqrt{2} + \sinh^{-1} 1]$$

$$= a\, [\sqrt{2} + \log (1 + \sqrt{1 + 1})] \qquad \left[\because \ \sinh^{-1} x = \log (x + \sqrt{x^2 + 1}) \right]$$

$$= a\, [\sqrt{2} + \log (1 + \sqrt{2})].$$

(*ii*) Length of arc L′OL cut off by the latus rectum = 2 times arc OL = $2a\, [\sqrt{2} + \log (1 + \sqrt{2})]$

Note 1. $\int \sqrt{\mathbf{a^2 - x^2}}\, \mathbf{dx} = \frac{\mathbf{x} \sqrt{\mathbf{a^2 - x^2}}}{\mathbf{2}} + \frac{\mathbf{a^2}}{\mathbf{2}}\, \mathbf{\sin^{-1} \frac{x}{a}}$.

2. $\int \sqrt{\mathbf{x^2 - a^2}}\, \mathbf{dx} = \frac{\mathbf{x} \sqrt{\mathbf{x^2 - a^2}}}{\mathbf{2}} - \frac{\mathbf{a^2}}{\mathbf{2}}\, \mathbf{\cosh^{-1} \frac{x}{a}}$.

3. $\int \sqrt{\mathbf{a^2 + x^2}}\, \mathbf{dx} = \frac{\mathbf{x} \sqrt{\mathbf{a^2 + x^2}}}{\mathbf{2}} + \frac{\mathbf{a^2}}{\mathbf{2}}\, \mathbf{\sinh^{-1} \frac{x}{a}}$.

Example 3. *Show that the length of an arc of the curve $x^2 = a^2 (1 - e^{y/a})$ measured from (0, 0) to (x, y) is $a \log \frac{a + x}{a - x} - x$.*

Sol. The given curve is $x^2 = a^2(1 - e^{y/a})$.

Let us solve for y in terms of x.

$$\therefore \quad x^2 = a^2 - a^2 e^{y/a} \quad \text{or} \quad a^2 e^{y/a} = a^2 - x^2 \quad \therefore \quad e^{y/a} = \frac{a^2 - x^2}{a^2}$$

$$\therefore \quad \frac{y}{a} = \log \left(\frac{a^2 - x^2}{a^2} \right) \text{ or } y = a \log \frac{a^2 - x^2}{a^2}$$

$$\therefore \quad \frac{dy}{dx} = a \,.\, \frac{a^2}{a^2 - x^2} \left(- \frac{2x}{a^2} \right) = - \frac{2ax}{a^2 - x^2}$$

$\therefore$ Required length of the arc from (0, 0) to (x, y) is

$$= \int_0^x \sqrt{1 + \left(\frac{dy}{dx} \right)^2}\, dx = \int_0^x \sqrt{1 + \frac{4a^2 x^2}{(a^2 - x^2)^2}}\, dx$$

$$= \int_0^x \sqrt{\frac{(a^2 - x^2)^2 + 4a^2 x^2}{(a^2 - x^2)^2}}\, dx = \int_0^x \frac{a^2 + x^2}{a^2 - x^2}\, dx$$

NOTES

$$= \int_0^x \left(-1 + \frac{2a^2}{a^2 - x^2} \right) dx$$

$$\begin{array}{r} -1 \\ -x^2 + a^2 \,\overline{)\, x^2 + a^2} \\ \underline{x^2 - a^2} \\ 2a^2 \end{array}$$

$$= \left[-x + 2a^2 \cdot \frac{1}{2a} \log \frac{a+x}{a-x} \right]_0^x \qquad \left| \because \int \frac{1}{\mathbf{a^2 - x^2}} \mathbf{dx} = \frac{1}{\mathbf{2a}} \mathbf{log} \frac{\mathbf{a+x}}{\mathbf{a-x}} \right.$$

$$= \left(-x + a \log \frac{a+x}{a-x} \right) - (-0 + a \log 1) = a \log \frac{a+x}{a-x} - x. \qquad [\because \log 1 = 0]$$

Note. $\int \frac{1}{\mathbf{x^2 - a^2}} \mathbf{dx} = \frac{1}{\mathbf{2a}} \mathbf{log} \frac{\mathbf{x-a}}{\mathbf{x+a}}$.

Example 4. *Show that the whole length of the curve* $x^2 (a^2 - x^2) = 8a^2 y^2$ *is* $\pi a \sqrt{2}$.

Sol. Equation of the curve is $8a^2 y^2 = x^2(a^2 - x^2)$ ...(1)

Let us trace the curve roughly to find the limits of integration.

(*i*) The curve is symmetrical about both the axes.

($\because$ There are only even powers of both x and y)

(*ii*) The curve passes through the origin and the tangents at the origin are given by $a^2x^2 = 8a^2 y^2$ or $y = \pm \frac{1}{2\sqrt{2}} x$ showing that origin is a node.

(*iii*) The curve has no asymptotes.

(*iv*) To find points of intersection with the x-axis *i.e.* with the line $y = 0$ [Putting $y = 0$ in (1)], we have $x^2 (a^2 - x^2) = 0$ or $x = 0, \pm a$.

Hence the curve meets the x-axis at (0, 0), (a, 0) and ($-a$, 0).

The curve meets the y-axis at the origin only.

(*v*) *Region*

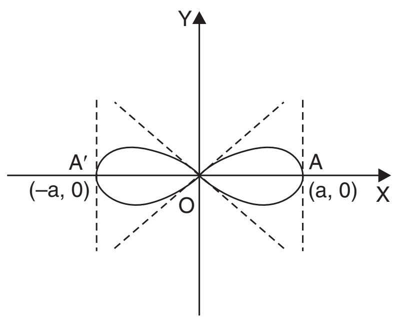
From (1), $$y = \frac{1}{\sqrt{8a^2}} x \sqrt{a^2 - x^2} \qquad ...(2)$$

For y to be real, $a^2 - x^2 \geq 0$ *i.e.* $x^2 \leq a^2$.

$\therefore \qquad -a \leq x \leq a.$

Thus, the whole curve lies between the lines $x = -a$ and $x = a$.

The shape of the curve is as shown in the figure. The curve consists of two symmetrical loops.



From (1), $$y = \frac{1}{\sqrt{8a^2}} x \sqrt{a^2 - x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{8a^2}} \left[x \left(\frac{-2x}{2\sqrt{a^2 - x^2}} \right) + \sqrt{a^2 - x^2} \right]$$

$$= \frac{1}{\sqrt{8a^2}} \left[\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} \right]$$

$\therefore$ Required whole length of the curve = 4 (Arc OA)

$$= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 4\int_0^a \sqrt{1+\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{8a^4-8a^2x^2+a^4+4x^4-4a^2x^2}{8a^2(a^2-x^2)}}\, dx$$

$$= \frac{4}{\sqrt{8a^2}}\int_0^a \sqrt{\frac{9a^4-12a^2x^2+4x^4}{a^2-x^2}}\, dx = \frac{4}{2a\sqrt{2}}\int_0^a \frac{3a^2-2x^2}{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\sqrt{2}}{a}\int_0^a \frac{2(a^2-x^2)+a^2}{\sqrt{a^2-x^2}}\, dx \qquad \textit{(Note this step)}$$

$$= \frac{\sqrt{2}}{a}\int_0^a \left(2\sqrt{a^2-x^2}+\frac{a^2}{\sqrt{a^2-x^2}}\right) dx$$

$$= \frac{\sqrt{2}}{a}\left[2\left(x\,\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)+a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\sqrt{2}}{a}\left[x\sqrt{a^2-x^2}+2a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\sqrt{2}}{a}\,[2a^2\sin^{-1}1] = \frac{\sqrt{2}}{a}\,.\,2a^2\,.\,\frac{\pi}{2} = \pi a\sqrt{2}.$$

Note. It is a common mistake not to take into account the symmetry of the curve. So the student is suggested to make use of the symmetry of the curve while writing limit of integration.

EXERCISE 5.1

1. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \dfrac{\pi}{3}$.

2. Find the length of the arc of the curve $y = \log \dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

$\Big[$ **Hint.** $\displaystyle\int \frac{e^{2x}+1}{e^{2x}-1}\,dx = \int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$ (Multiplying every term by e^{-x})

$= \log(e^x - e^{-x})$. $\left(\because \displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right)\Big]$

3. (*i*) Find the length of the curve $3ay^2 = 2x^3$ between the points (0, 0) and (x, y).

(*ii*) Find the length of the curve $ay^2 = x^3$ from the vertex to the point (a, a).

[**Hint.** Vertex is (0, 0)].

4. Find the length of the arc of the parabola $x^2 = 4ay$:

(*i*) from the vertex to an extremity of the latus rectum.

(*ii*) cut off by latus rectum.

5. (*i*) Find the arc length of the curve

$$y = \frac{1}{2}x^2 - \frac{1}{4}\log x \text{ from } x = 1 \text{ to } x = 2.$$

(*ii*) Find the length of the arc of the curve $y = x(2 - x)$ as x varies from 0 to 2.

6. (*i*) Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$.

$$\left[\textbf{Hint. } \int \frac{x-a}{\sqrt{x-2a}}\,dx = \int \frac{x-2a+a}{\sqrt{x-2a}}\,dx.\right]$$

NOTES

(*ii*) Find the perimeter of the loop of the curve $9y^2 = x^3 - 12x^2 + 45x - 50$.

[**Hint.** $x^3 - 12x^2 + 45x - 50$

$= x^3 - 2x^2 - 10x^2 + 20x + 25x - 50$

$= x^2(x-2) - 10x(x-2) + 25(x-2) = (x-2)(x^2 - 10x + 25)$

$= (x-2)(x-5)^2$.

Now it is part (*i*) with $a = 1$].

7. Find the whole length of the loop of the curve $3ay^2 = x(x-a)^2$.

8. Rectify the loop of the curve $3ay^2 = x^2(a - x)$.

9. Find the whole length of the loop of the curve $3ax^2 = y^2(a - y)$.

10. Find the whole length of the loop of the curve $3ax^2 = y(y-a)^2$.

11. Find the length of the arc of the curve $x^2 + y^2 - 2ax = 0$ in first quadrant.

Answers

1. $\log(2+\sqrt{3})$ **2.** $\log(e + e^{-1})$

3. (*i*) $\frac{4a}{9}\left[\left(1+\frac{3x}{2a}\right)^{3/2} - 1\right]$ (*ii*) $\frac{a}{27}[13\sqrt{13} - 8]$

4. (*i*) $a[\sqrt{2} + \log(\sqrt{2}+1)]$ (*ii*) $2a[\sqrt{2} + \log(\sqrt{2}+1)]$

5. (*i*) $\frac{3}{2} + \frac{1}{4}\log 2$ (*ii*) $\frac{1}{2}\log(2+\sqrt{5}) + \sqrt{5}$

6. (*i*) $4\sqrt{3}a$ (*ii*) $4\sqrt{3}$ **7.** $\frac{4a}{\sqrt{3}}$ **8.** $\frac{4a}{\sqrt{3}}$

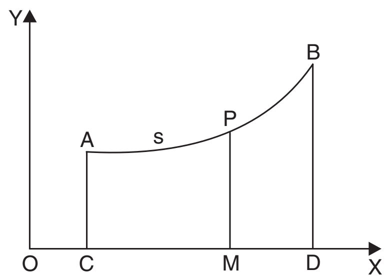
9. $\frac{4a}{\sqrt{3}}$ **10.** $\frac{4a}{\sqrt{3}}$ **11.** πa.

LENGTH OF THE CURVE WHEN PARAMETRIC EQUATIONS ARE GIVEN

To prove that the length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = t_1$ to $t = t_2$ is given by

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

where x and y are continuous and single-valued functions of t, in the given interval $[t_1, t_2]$.



Let AB be the curve $x = f(t)$, $y = \phi(t)$ where A, B are two points where $t = t_1$ and $t = t_2$ and let CA and DB be their ordinates.

Let P (x, y) be any point on the curve and PM be its ordinate.

If s is the length of the arc from the fixed point A to the variable point P, then clearly s is a function of t.

From Differential Calculus, we know that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \ldots(1)$$

$$\therefore \quad \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t_1}^{t_2} \frac{ds}{dt}\,.\,dt = \Big[s\Big]_{t_1}^{t_2}$$

$$= (\text{value of } s \text{ when } t = t_2) - (\text{value of } s \text{ when } t = t_1)$$

$$= \text{arc AB} - 0 = \text{arc AB}.$$

Hence arc AB $= \displaystyle\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

Note. Here again it is assumed that the integrand does not change sign in the range of integration $[t_1, t_2]$.

Cor. From (1), $\quad s = \displaystyle\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

Example 1. *Find the length of the curve*

$$x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right),\ y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$$

measured from $\theta = 0$ *to* $\theta = \pi$.

Sol. The given curve is

$$x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right), \quad y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$$

$$\therefore \quad \frac{dx}{d\theta} = e^{\theta}\left(\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) + e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right) = \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

and

$$\frac{dy}{d\theta} = e^{\theta}\left(-\frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) + e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right) = -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{25}{4}e^{2\theta}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{25}{4}e^{2\theta} \qquad \ldots(1)$$

$$\therefore \quad \text{Required Length} = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \frac{5}{2}e^{\theta}\, d\theta \qquad |\ \text{Using (1)}$$

$$= \frac{5}{2}\Big[e^{\theta}\Big]_0^{\pi} = \frac{5}{2}[e^{\pi} - 1].$$

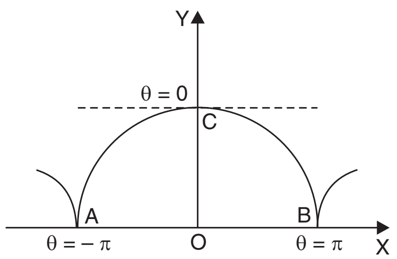
Example 2. *Rectify the curve* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$.

Sol. The equations of the cycloid are $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ $\qquad \ldots(1)$

We trace this cycloid for values of θ in $[-\pi, \pi]$ to get one **arch** of the cycloid.

This curve is symmetrical about y-axis.

($\because$ On changing θ to $-\theta$, x changes to $-x$ and y is unchanged)



NOTES

The shape of the complete cycloid is being shown below :

From (1), $\frac{dx}{d\theta} = a\,(1 + \cos\theta)$ and $\frac{dy}{d\theta} = -\,a \sin\theta$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\,(1+\cos\theta)^2 + a^2 \sin^2\theta$$

$$= a^2\,[1 + \cos^2\theta + 2\cos\theta + \sin^2\theta]$$

$$= 2a^2\,(1+\cos\theta) = 4a^2\cos^2\frac{\theta}{2} \qquad \ldots(2)$$

∴ Length of the curve (*i.e.* length of *one arch of cycloid* or *complete cycloid*).

$$= 2\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 2\int_0^{\pi} \sqrt{4a^2\cos^2\frac{\theta}{2}}\, d\theta \qquad | \text{ From (2)}$$

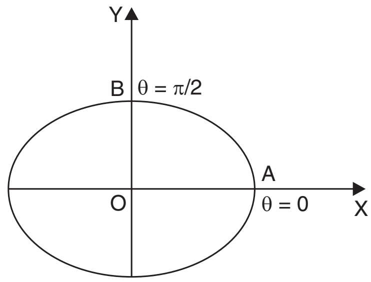
$$= 2\int_0^{\pi} 2a\cos\frac{\theta}{2}\, d\theta = 4a\left(\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right)_0^{\pi} = 8a\left[\sin\frac{\pi}{2} - \sin 0\right] = 8a.$$

Note. We have taken $\sqrt{4a^2\cos^2\theta/2} = 2a\cos\theta/2$ and not $-\,2a\cos\theta/2$ as $\cos\theta/2$ remains positive when θ increases 0 to π.

Example 3. *Rectify the ellipse $x = a\cos\theta$ and $y = b\sin\theta$.*

or $\left(\textit{Find the whole length of the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$.

Sol. The ellipse $x = a\cos\theta$, $y = b\sin\theta$, $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$ is symmetrical about both the axes.



Diff. w.r.t. θ, $\frac{dx}{d\theta} = -\,a\sin\theta$ and $\frac{dy}{d\theta} = b\cos\theta$

∴ Perimeter of the ellipse

$$= 4\,(\text{Arc AB})$$

$$= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2(1-e^2)\cos^2\theta}\, d\theta$$

[∵ For an ellipse $b^2 = a^2\,(1 - e^2)$, where e is the eccentricity of the ellipse]

$$= 4a\int_0^{\pi/2} \sqrt{(\sin^2\theta + \cos^2\theta) - e^2\cos^2\theta}\, d\theta$$

$$= 4a\int_0^{\pi/2} (1 - e^2\cos^2\theta)^{1/2}\, d\theta$$

NOTES

$$= 4a \int_0^{\pi/2} \left[1 + \frac{1}{2}(-e^2\cos^2\theta) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-e^2\cos^2\theta)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-e^2\cos^2\theta)^3 + \ldots \right] d\theta$$

$$\left[\because \text{ By Binomial Theorem, } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\right]$$

$$= 4a \int_0^{\pi/2} \left[1 - \frac{1}{2}e^2\cos^2\theta - \frac{1}{2.4}e^4\cos^4\theta - \frac{1.3}{2.4.6}e^6\cos^6\theta - \ldots\right] d\theta$$

$$= 4a\left[\int_0^{\pi/2} 1\,d\theta - \frac{1}{2}e^2\int_0^{\pi/2}\cos^2\theta\,d\theta - \frac{1}{2.4}e^4\int_0^{\pi/2}\cos^4\theta\,d\theta - \frac{1.3}{2.4.6}e^6\int_0^{\pi/2}\cos^6\theta\,d\theta - \ldots\right]$$

$$= 4a\left[\frac{\pi}{2} - \frac{1}{2}e^2 \cdot \frac{1}{2}\frac{\pi}{2}^* - \frac{1}{2.4}e^4 \cdot \frac{3.1}{4.2}\frac{\pi}{2} - \frac{1.3}{2.4.6}e^6 \cdot \frac{5.3.1}{6.4.2}\frac{\pi}{2} - \ldots\right]$$

$$= 2a\pi\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1.3}{2.4}\right)^2\frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\frac{e^6}{5} - \ldots\right]$$

EXERCISE 5.2

1. Find the length of the arc of the curve $x = e^\theta \sin\theta$, $y = e^\theta\cos\theta$ from $\theta = 0$ to be $\theta = \frac{\pi}{2}$.

2. Find the length of the complete cycloid given by

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta).$$

[**Hint.** For values of θ in $[-\pi, \pi]$, we get the complete cycloid. Length of complete cycloid is

$$2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.]$$

3. (*i*) Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

(*ii*) and show that $\theta = \frac{2\pi}{3}$ divides it in the ratio 1 : 3.

[**Hint.** For values of θ in $[0, 2\pi]$, we get the complete cycloid.

Length of complete cycloid is $2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.$]

$$* \int_0^{\pi/2}\cos^n\theta\,d\theta = \frac{(n-1)\times \text{go on decreasing by 2}}{n \times \text{go on decreasing by 2}} \times \left(\frac{\pi}{2} \text{ if } n \text{ is an even positive integer}\right)$$

NOTES

4. Rectify the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 + \cos\theta)$.

[**Hint.** Same as for question 3.]

5. Show that the entire length of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ (or $x^{2/3} + y^{2/3} = a^{2/3}$ or $(x/a)^{2/3} + (y/a)^{2/3} = 1$) is $6a$.

$$\left[\textbf{Hint}.\ \text{Entire Length} = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.\right]$$

6. Find the length of the arc in the first quadrant of the curve

$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ or ($x = a\cos^3 t$, $y = b\sin^3 t$).

[**Hint.** Length of arc in **first** quadrant

$$= \int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\pi/2} 3\sin t\cos t\,\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt.$$

[Put $a^2\cos^2 t + b^2\sin^2 t = z$.]

7. Prove that the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ is of length $4\sqrt{3}$.

[**Hint.** Trace this curve. Limits of integration are 0 to 3. The curve is symmetrical about x-axis.]

8. Prove that in the cycloid $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$; the length of the arc from the vertex to the point (x, y) is $\sqrt{8ay}$. Hence find the whole length of the curve.

9. (*i*) Find the length of the arc of the curve.

$x = a\,(\cos t + t\sin t)$, $y = a\,(\sin t - t\cos t)$ from $t = 0$ to $t = 2$.

(*ii*) A curve is given by the equation $x = a\,(\cos\theta + \theta\sin\theta)$, $y = a\,(\sin\theta - \theta\cos\theta)$. Find the length of the arc from $\theta = 0$ to $\theta = \alpha$.

10. Show that the length of an arc of the curve $x\sin\theta + y\cos\theta = f'(\theta)$, $x\cos\theta - y\sin\theta = f''(\theta)$ is given by $s = f(\theta) + f''(\theta) + c$.

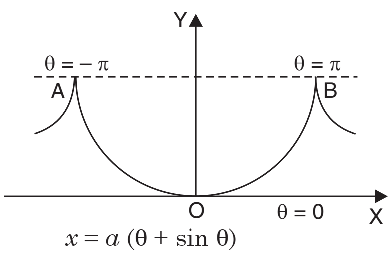
[**Hint.** Solving the two equations for x and y, we have $x = f'(\theta)\sin\theta + f''(\theta)\cos\theta$, $y = f'(\theta)\cos\theta - f''(\theta)\sin\theta$.]

Answers

1. $\sqrt{2}\,(e^{\pi/2} - 1)$	**2.** $8a$	**3.** $8a$	**4.** $8a$
6. $\frac{a^2 + ab + b^2}{a + b}$	**8.** $8a$	**9.** (*i*) $2a$	(*ii*) $\frac{a\alpha^2}{2}$

Remark. For the sake of convenience of the readers, we are giving below the shapes of the curves in questions **2, 3, 4, 5, 6, 7.**

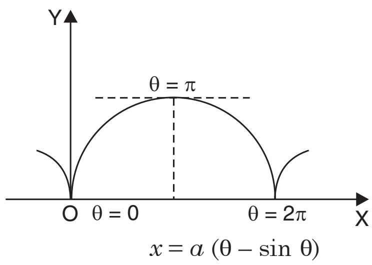
2.



$x = a\,(\theta + \sin\theta)$

$y = a\,(1 - \cos\theta)$

3.

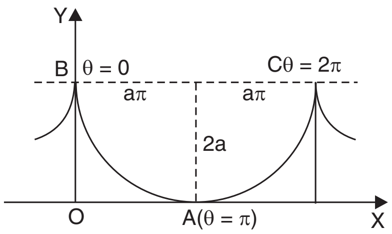


$x = a\,(\theta - \sin\theta)$

$y = a\,(1 - \cos\theta)$

NOTES

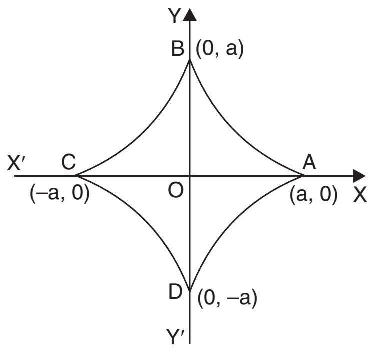
4.



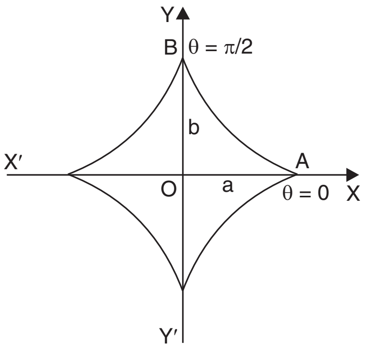
$x = a\,(\theta - \sin\theta)$

$y = a\,(1 + \cos\theta)$

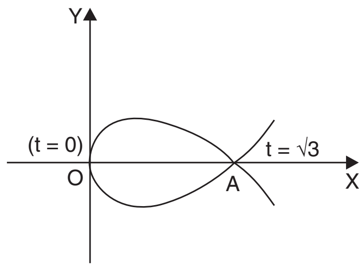
5.



6.



7.



LENGTH OF THE POLAR CURVES

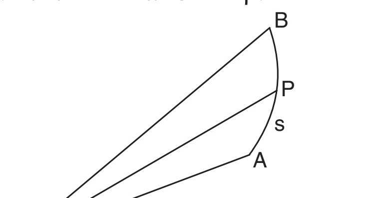
To prove that the length of the arc of the curve r = f(θ) between points whose vectorial angles are α and β, is

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

where $\frac{dr}{d\theta}$ *is continuous and single valued in* $[\alpha, \beta]$.

Let AB be the curve $r = f(\theta)$ and A, B the points where $\theta = \alpha$ and $\theta = \beta$.

Let $P(r, \theta)$ be *any point* on the curve and let s denotes the length of arc AP which is clearly a function of θ.



From Differential Calculus, we know that

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad \ldots(1)$$

$$\therefore \quad \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \frac{ds}{d\theta} \cdot d\theta = \Big[s\Big]_\alpha^\beta$$

= [Value of s when $\theta = \beta$] – [Value of s when $\theta = \alpha$] = arc AB – 0.

$$\therefore \quad \text{arc AB} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

NOTES

Cor. 1. From (1), $\mathbf{s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.}$

Cor. 2. *When the equation of the curve is of the form* $\theta = \mathbf{f(r)}$, *the length of the arc between the two points whose radii vectors are* r_1 *and* r_2 *is*

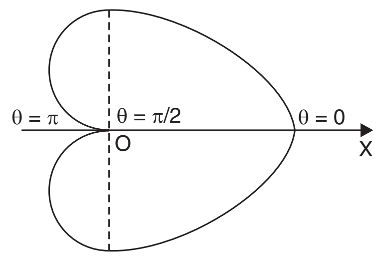
$$\int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr.$$

where $\frac{d\theta}{dr}$ *is continuous and single-valued in* $[r_1, r_2]$.

Proceed as in Art. 4, above and use $\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$ instead of (1).

Example 1. *Find the entire length of the cardioid* $r = a\,(1 + \cos\theta)$, *and show that the arc of the upper half is bisected by* $\theta = \pi/3$.

Sol. The equation of the curve is $r = a(1 + \cos\theta)$. ...(1)



The shape of the cardioid is as shown in the adjoining figure. It is symmetrical about the initial line and the upper half of the curve is traced when θ varies from 0 to π.

From (1), $\frac{dr}{d\theta} = -a\sin\theta$

∴ Length of whole arc = 2 × length of upper half

$$= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = 2a\int_0^{\pi} 2\cos\theta/2\, d\theta$$

$$= 4a\int_0^{\pi} \cos\theta/2\, d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a(1-0) = 8a.$$

∴ Entire length of cardioid = $8a$ and length of the upper half = $4a$.

Now the length of arc from θ = 0 to θ* = π/3

$$= \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/3} 2a\cos\theta/2\, d\theta$$

$$= 2a\left|2\sin\frac{\theta}{2}\right|_0^{\pi/3} = 4a\left|\frac{1}{2} - 0\right| = 2a$$

$$= \frac{1}{2} \times \text{length of the upper half arc.}$$

Hence the length of the upper half arc is bisected at θ = π/3.

*θ = α is the equation of a half ray passing through the pole and making an angle α with the initial line.

NOTES

Example 2. *Show that the arc of the hyperbolic spiral r θ = a taken from the point*

$$r = a \text{ to } r = 2a \text{ is } a\left[\sqrt{5} - \sqrt{2} + \log\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right)\right].$$

Sol. The equation of the given curve is $r\theta = a$.

$$\therefore \qquad \theta = \frac{a}{r} \quad \therefore \quad \frac{d\theta}{dr} = -\frac{a}{r^2}$$

∴ Required length from $r = a$ to $r = 2a$ is given by

$$s = \int_a^{2a} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr \qquad \text{(Cor. 2, Art. 4)}$$

$$= \int_a^{2a} \sqrt{1 + r^2 \cdot \frac{a^2}{r^4}}\, dr = \int_a^{2a} \frac{\sqrt{r^2 + a^2}}{r}\, dr$$

Put $\sqrt{\mathbf{r^2 + a^2}} = \mathbf{t}$ $\quad\therefore\quad r^2 + a^2 = t^2$

∴ $2r\,dr = 2t\,dt$ or $r\,dr = t\,dt$.

When $r = a$, $t = a\sqrt{2}$ and when $r = 2a$, $t = a\sqrt{5}$

$$\therefore \qquad s = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t}{t^2 - a^2} \cdot t\, dt = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t^2}{t^2 - a^2}\, dt$$

$$= \int_{a\sqrt{2}}^{a\sqrt{5}} \left(1 + \frac{a^2}{t^2 - a^2}\right) dt = \left[t + a^2 \cdot \frac{1}{2a}\log\frac{t-a}{t+a}\right]_{a\sqrt{2}}^{a\sqrt{5}}$$

$$= a\sqrt{5} + \frac{a}{2}\log\frac{a\sqrt{5} - a}{a\sqrt{5} + a} - \left(a\sqrt{2} + \frac{a}{2}\log\frac{a\sqrt{2} - a}{a\sqrt{2} + a}\right)$$

$$= a\left[\sqrt{5} + \frac{1}{2}\log\frac{\sqrt{5}-1}{\sqrt{5}+1} - \sqrt{2} - \frac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + \frac{1}{2}\left(\log\frac{\sqrt{5}-1}{\sqrt{5}+1} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + \frac{1}{2}\log\left(\frac{(\sqrt{5}-1)}{(\sqrt{5}+1)}\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}\right)\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + \frac{1}{2}\log\left(\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{(\sqrt{5}+1)^2}\frac{(\sqrt{2}+1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}\right)\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + \frac{1}{2}\log\frac{4(\sqrt{2}+1)^2}{(\sqrt{5}+1)^2}\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + \frac{1}{2}\log\left(\frac{2(\sqrt{2}+1)}{\sqrt{5}+1}\right)^2\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + 2 \cdot \frac{1}{2}\log\frac{\sqrt{8}+2}{\sqrt{5}+1}\right]$$

$$= a\left[\sqrt{5} - \sqrt{2} + \log\left(\frac{\sqrt{8}+2}{\sqrt{5}+1}\right)\right].$$

NOTES

EXERCISE 5.3

1. Find the length of the arc of the cardioid $r = a\ (1 - \cos\theta)$ between the points whose vectorial angles are α and β.
2. Find the perimeter of the curve $r = a\ (1 - \cos\theta)$ and show that the arc of the upper half of the curve $r = a\ (1 - \cos\theta)$ is bisected by $\theta = \dfrac{2\pi}{3}$.
3. Find the perimeter of the curve (circle) $r = a \cos\theta$.

   $\left[\textbf{Hint. } s = 2\int_0^{\pi/2} \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta.\right]$
4. Find the length of a loop of the curve $r = a(\theta^2 - 1)$.

   [**Hint.** To find the limits of integration for a loop, we generally put $r = 0$ and find two consecutive values of θ. Here putting $r = 0$, $\theta = \pm 1$.

   $\therefore$ Limits of integration are -1 and 1.]
5. Find the length of the arc of the equiangular spiral $r = ae^{\theta \cot \alpha}$ between the points for which radii vectors are r_1 and r_2.
6. Find the length of the arc of the spiral $r = a\theta$ between the points whose radii vectors are r_1 and r_2.
7. (*a*) Prove that the cardioide $r = a\ (1 + \cos\theta)$ is divided by the line $4r \cos\theta = 3a$ into two parts such that the lengths of the arcs on either side of this line are equal.

   [**Hint.** Solve $r = a\ (1 + \cos\theta)$ and $4r\cos\theta = 3a$ for r and θ to find the points of intersection.]

   (*b*) Show that the length of the arc of that part of the cardioid $r = a(1 + \cos\theta)$ which lies on the side of the line $4r = 3a \sec\theta$ remote from the pole, is equal to $4a$.
8. Find the length of the arc of the parabola $\dfrac{2a}{r} = 1 + \cos\theta$ cut off by its latus rectum.

   $\left[\textbf{Hint. } \text{The equation of latus rectum is } \theta = \dfrac{\pi}{2}.\right]$
9. Prove that the perimeter of the limicon $r = a + b\cos\theta$ if $\dfrac{b}{a}$ is small, is approximately $2\pi\ a \left[1 + \dfrac{1}{4} \cdot \dfrac{b^2}{a^2}\right]$.
10. Show that the whole length of the limicon $r = a + b\cos\theta$ $(a > b)$ is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limicon.

    $\left[\textbf{Hint. } \text{Length of the limicon} = 2\int_0^{\pi} \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta.\right.$

    Maximum and minimum radii vectors of limicon are $a + b$ and $(a - b)$.

    $\therefore$ Equations of the ellipse are $x = (a + b)\cos t$, $y = (a - b)\sin t$.

    $\left.\text{Whole length of the ellipse is } 4\int_0^{\pi/2} \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt.\right]$

Answers

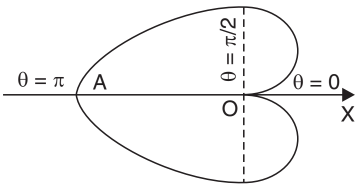
1. $4a\left(\cos\dfrac{\alpha}{2} - \cos\dfrac{\beta}{2}\right)$
2. $8a$
3. πa
4. $\dfrac{8a}{3}$
5. $(r_2 - r_1)\sec\alpha$

6. $\dfrac{1}{2a}$ $[f(r_2) - f(r_1)]$ where $f(r) = r\sqrt{r^2 + a^2} + a^2 \sinh^{-1}\dfrac{r}{a}$

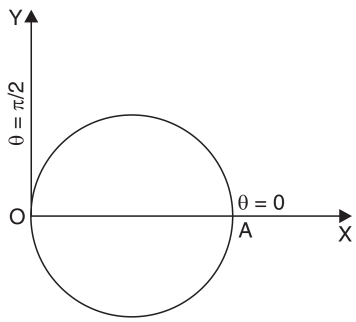
8. $2a\,[\sqrt{2} + \log(\sqrt{2} + 1)]$.

Remark. For the sake of convenience of the readers, we are giving below the shapes of the curves in Questions **2, 3, 8, 9** and **10.**

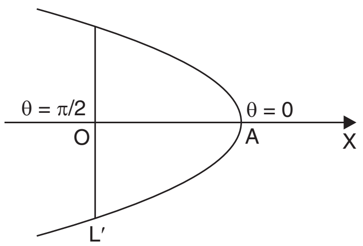
2.



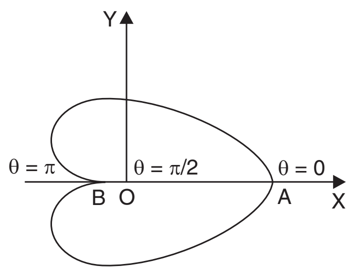
3.



8.



9. and 10.



$r = a(1 - \cos\theta)$

TO PROVE THAT THE LENGTH OF THE ARC OF THE CURVE p = f(r) BETWEEN THE POINTS WHERE r = a, r = b IS

$$\int_a^b \frac{\mathbf{r}}{\sqrt{\mathbf{r}^2 - \mathbf{p}^2}}\,\mathbf{dr}.$$

From Cor. 2. Art. 4,

$$\text{Length of arc} = \int_a^b \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr = \int_a^b \sqrt{1 + \tan^2\phi}\,dr \qquad \left(\because\ \mathbf{\tan\phi = r\,\frac{d\theta}{dr}}\right)$$

$$= \int_a^b \sec\phi\,dr = \int_a^b \frac{r}{\sqrt{r^2 - p^2}}\,dr$$

$$\left(\because\ \mathbf{p = r\sin\phi}\ \ \therefore\ \sec\phi = \frac{1}{\cos\phi} = \frac{1}{\sqrt{1 - \sin^2\phi}} = \frac{1}{\sqrt{1 - \dfrac{p^2}{r^2}}} = \frac{r}{\sqrt{r^2 - p^2}}\right)$$

[**Note. p** mentioned in the above Article is the length of the perpendicular from the pole on **any** tangent and ϕ is the angle between the tangent and radius vector at any point.]

NOTES

EXERCISE 5.4

1. Show that the length of the arc of the equiangular spiral $p = r \sin \alpha$ between the points at which radii vectors are r_1 and r_2 is $(r_2 - r_1) \sec \alpha$.
2. Show that the length of the arc of the hyperbola $xy = a^2$ between the limits $x = b$ and $x = c$ is equal to the arc of the curve $p^2 (a^4 + r^4) = a^4 r^2$ between the limits $r = b$, $r = c$.

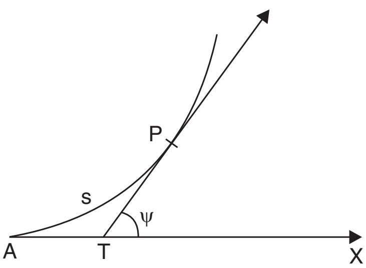
[**Hint**. Equations of two curves are $y = \dfrac{a^2}{x}$ and $p^2 = \dfrac{a^4 r^2}{a^4 + r^4}$.

Do not evaluate the two integrals. Show that they are equal.]

INTRINSIC EQUATION OF A CURVE

A relation between the **variables** s and ψ is called **intrinsic equation** of a curve.

s is the length of the arc of a curve measured from a fixed point A on it to **ANY** point P and ψ is the angle which the tangent at P makes with the tangent at A (or with any other fixed line generally x-axis in the plane of the curve).



[**Note.** A relation between x and y is called *Cartesian Equation*.

A relation between r and θ is called *Polar Equation*.

A relation between p and r is called *Pedal Equation*.]

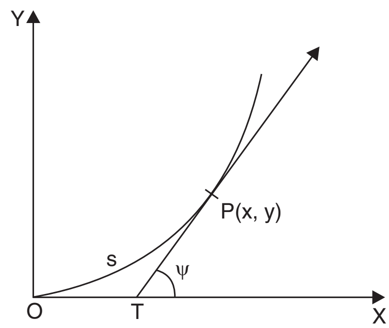
TO FIND THE INTRINSIC EQUATION OF A CURVE FROM THE CARTESIAN EQUATION

Let $y = f(x)$ be *cartesian equation* of curve, where x-axis is the tangent at the origin O, the fixed point from which arc s(= OP) is measured. We know that

$$\tan \psi = \frac{dy}{dx} = f'(x) \qquad ...(1)$$

and
$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = F(x) \text{ (say)} \qquad ...(2)$$

Eliminating x between (1) and (2), we get a relation between s and ψ which is the intrinsic equation of the curve.

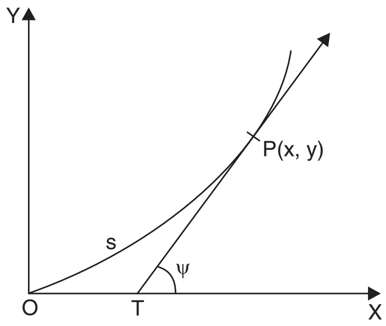


NOTES

TO FIND THE INTRINSIC EQUATION OF THE CURVE FROM THE PARAMETRIC EQUATIONS

Let $x = f(t)$, $y = F(t)$, be the parametric equations of the curve and let the x-axis be the fixed straight line (tangent at the origin from which the arc is measured).

Let P(x, y) be any point on the curve and PT the tangent at P meeting the x-axis in T. Let arc OP = s and $\angle$XTP = ψ.



Now $\tan \psi = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{F'(t)}{f'(t)}$ ...(1)

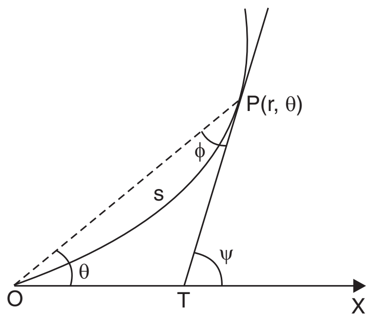
and $$s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^t \sqrt{[f'(t)]^2 + [F'(t)]^2}\, dt = \phi\,(t),\ (say) \qquad ...(2)$$

Eliminating t between (1) and (2), we obtain a relation between s and ψ which is the intrinsic equation of the curve.

TO FIND THE INTRINSIC EQUATION OF THE CURVE FROM POLAR EQUATION

Let $r = f(\theta)$ be the polar equation of the curve, the initial line being the tangent to the curve at the pole O, the fixed point, from which arc is measured. Let P(r, θ) be any point on the curve, so that arc OP = s. Let tangent at P meet the initial line in T, so that $\angle$XTP = ψ.



Now $\psi = \theta + \phi$ ...(1)

$$\tan \phi = r\frac{d\theta}{dr} = \frac{f(\theta)}{f'(\theta)} \qquad ...(2)$$

and $$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

$$= F(\theta) \qquad (say) \qquad ...(3)$$

Eliminating θ and ϕ between (1), (2) and (3), we get a relation between s and ψ, which is the intrinsic equation of the curve.

TO FIND THE INTRINSIC EQUATION OF THE CURVE FROM THE PEDAL EQUATION

Let $p = f(r)$ be the pedal equation of the curve, the pole being at O, a fixed point on the curve and the initial line coinciding with the tangent at O, so here p and r vanish simultaneously at O.

NOTES

We know that

$$\frac{\mathbf{ds}}{\mathbf{d\psi}} = \rho = \mathbf{r}\,\frac{\mathbf{dr}}{\mathbf{dp}} = \frac{r}{dp/dr} = \frac{r}{f'(r)} \qquad ...(1)$$

Also $$s = \int_0^r \frac{r\,dr}{\sqrt{r^2 - p^2}} = \int_0^r \frac{r\,dr}{\sqrt{r^2 - [f(r)]^2}} = f_1(r) \quad (say) \qquad ...(2)$$

Eliminating r from (1) and (2), we get a relation between s and $\frac{ds}{d\psi}$ say $\frac{ds}{d\psi}$ $= \mathrm{F}(s)$ or $\frac{d\psi}{ds} = \frac{1}{\mathrm{F}(s)}$.

$\therefore \quad \psi = \int \frac{ds}{\mathrm{F}(s)}$ which on integration gives us a relation between ψ and s *i.e.* the intrinsic equation of the curve.

Example 1. *Find the intrinsic equation of the semi-cubical parabola, $ay^2 = x^3$, taking the cusp as the fixed point.*

Sol. The equation of the semi-cubical parabola is $ay^2 = x^3$ ...(1)

Differentiating $2ay \cdot \frac{dy}{dx} = 3x^2$ or $\frac{dy}{dx} = \frac{3x^2}{2ay}$.

Step I. Measuring s from the cusp (*at the origin, where $x = 0$*),

we have $$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{1 + \frac{9x^4}{4a^2y^2}}\,dx$$

$$= \int_0^x \sqrt{1 + \frac{9x^4}{4ax^3}}\,dx = \int_0^x \sqrt{1 + \frac{9x}{4a}}\,dx$$

$$= \left| \frac{\left(1 + \frac{9x}{4a}\right)^{3/2}}{\frac{3}{2} \cdot \frac{9}{4a}} \right|_0^x = \frac{8a}{27}\left[\left(1 + \frac{9x}{4a}\right)^{3/2} - 1\right] \qquad ...(2)$$

Step II. $$\tan\psi = \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\therefore \qquad \tan^2\psi = \frac{9x^4}{4a^2y^2} = \frac{9x^4}{4ax^3} = \frac{9x}{4a} \qquad ...(3)$$

Step III. Eliminating x from (2) and (3), we get

$$s = \frac{8a}{27}\,[(1 + \tan^2\psi)^{3/2} - 1] = \frac{8a}{27}\,[\sec^3\psi - 1]$$

or $$27s = 8a\,(\sec^3\psi - 1).$$

this is the required intrinsic equation.

Example 2. *Find the intrinsic equation of the cycloid $x = a\,(t + \sin t)$, $y = a\,(1 - \cos t)$ and prove that $s^2 + \rho^2 = 16\,a^2$.*

Sol. The equations of the cycloid are $x = a(t + \sin t)$, $y = a(1 - \cos t)$.

$$\therefore \qquad \frac{dx}{dt} = a\,(1 + \cos t),\ \frac{dy}{dt} = a\sin t$$

Step I. Measuring s from the point where $t = 0$,

$$s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^t \sqrt{a^2(1+\cos t)^2 + a^2 \sin^2 t}\, dt$$

$$= a \int_0^t \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= a \int_0^t \sqrt{2(1+\cos t)}\, dt = a \int_0^t \sqrt{2.2\cos^2 \frac{t}{2}}\, dt = 2a \int_0^t \cos\frac{t}{2}\, dt$$

$$= 2a \left[\frac{\sin t/2}{\frac{1}{2}}\right]_0^t = 4a\left[\sin\frac{t}{2}\right] \qquad ...(1)$$

Step II. $\tan\psi = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a\sin t}{a(1+\cos t)} = \dfrac{2\sin t/2 . \cos t/2}{2\cos^2 t/2} = \tan t/2$

$\therefore \qquad \psi = t/2 \qquad ...(2)$

Step III. Eliminating t from (1) and (2) [by putting $t/2 = \psi$ from (2) in (1)], we get $s = 4a \sin\psi$. This is the required intrinsic equation.

Diff. w.r.t. ψ, $\dfrac{ds}{d\psi} = 4a\cos\psi$ or $\rho = 4a\cos\psi$

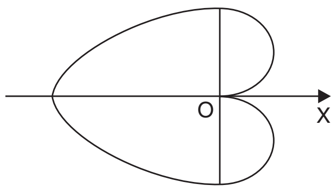
$$\therefore \quad s^2 + \rho^2 = 16a^2\sin^2\psi + 16a^2\cos^2\psi$$

$$= 16a^2(\sin^2\psi + \cos^2\psi) = 16a^2.$$

Example 3. *Find the intrinsic equation of the cardioid $r = a(1 - \cos\theta)$.*

Sol. Suppose s is measured from the pole, where (putting $r = 0$), we have $\theta = 0$. Now

$$\frac{dr}{d\theta} = a\sin\theta.$$



Step I.

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{2a^2(1-\cos\theta)}\, d\theta = \int_0^\theta \sqrt{2a^2 . 2\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^\theta 2a\sin\theta/2\, d\theta = -4a\left[\cos\frac{\theta}{2}\right]_0^\theta$$

$$= -4a[\cos\theta/2 - 1] = 4a(1 - \cos\theta/2)$$

$$= 4a . 2\sin^2\theta/4 = 8a\sin^2\theta/4 \qquad ...(1)$$

Step II. Now $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \tan\theta/2$

$\therefore \qquad \phi = \dfrac{\theta}{2} \qquad ...(2)$

Step III. Again $\psi = \theta + \phi = \theta + \theta/2 = 3\theta/2$ [Using (2)]

$\therefore$ $\theta = 2\psi/3$ or $\theta/4 = \psi/6$ ...(3)

Step IV. Eliminating θ from (1) and (3), we get

$$s = 8a \sin^2 \psi/6.$$

This is the required intrinsic equation of the curve.

Example 4. *Find the intrinsic equation of the curve whose pedal equation is*

$$p^2 = r^2 - a^2.$$

Sol. The given equation of the curve is $p^2 = r^2 - a^2$. ...(1)

$\therefore$ $$2p \frac{dp}{dr} = 2r \text{ or } p \frac{dp}{dr} = r \text{ or } \frac{p}{r} = \frac{dr}{dp}$$

$\therefore$ $$\frac{\mathbf{ds}}{\mathbf{d\psi}} = \rho = \mathbf{r} \frac{\mathbf{dr}}{\mathbf{dp}} = r \cdot \frac{p}{r} = p \quad \text{...(2)}$$

Step I. Let s be measured from the point **where r = a.** (*i.e.* $s = 0$)

[$\because$ For $r = 0$, $p^2 = -a^2$ from (1) and hence p is Imaginary]

Step II. $$s = \int_a^r \frac{rdr}{\sqrt{r^2 - p^2}} = \int_a^r \frac{r}{a} dr \quad [\because \text{ from (1), } r^2 - p^2 = a^2]$$

$$= \frac{1}{2a} \Big| r^2 \Big|_a^r = \frac{1}{2a}(r^2 - a^2) = \frac{p^2}{2a} \quad \text{...(3) [Using (1)]}$$

Step III. From (2) and (3), eliminating p, we get

[From (3), $p^2 = 2as$ $\therefore$ $p = \sqrt{2as}$. Putting this value of p in (2)]

$$\frac{ds}{d\psi} = \sqrt{2as}.$$

$\therefore$ $$\frac{ds}{\sqrt{s}} = \sqrt{2a}\, d\psi$$

Step IV. Integrating, we have $\int s^{-1/2}\, ds = \sqrt{2a} \int 1\, d\psi + k$

i.e. $$2\sqrt{s} = \sqrt{2a}\, \psi + k, \quad \text{...(4)}$$

where k is some constant of integration.

Let when $\psi = 0, s = 0$ $\therefore$ From (4), $k = 0$.

$\therefore$ From (4), $2\sqrt{s} = \sqrt{2a}\, \psi$

Squaring both sides, $4s = 2a\psi^2$ or $s = \frac{a}{2}\psi^2$.

This is the required intrinsic equation.

EXERCISE 5.5

1. Find the intrinsic equation of the catenary $y = c \cosh \frac{x}{c}$, s being measured from the vertex $(0, c)$ of the catenary and show that $c\rho = c^2 + s^2$, ρ being the radius of curvature.
2. Show that the intrinsic equation of the parabola $y^2 = 4ax$ is
$$s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi).$$

NOTES

3. Show that the intrinsic equation of the parabola $x^2 = 4ay$ is $s = a \tan \psi \sec \psi + a \log (\tan \psi + \sec \psi)$.
4. Find the intrinsic equation of the curve $y = a \log \sec \dfrac{x}{a}$, s being measured from the origin.
5. Find the intrinsic equation of the parabola $x = at^2$, $y = 2at$, these are being measured from the vertex.
6. Find the intrinsic equation of the curve $x = a(1 + \sin t)$, $y = a(1 + \cos t)$.
7. In the four-cusped hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$, show that

   (*i*) $s = \dfrac{3a}{4} \cos 2\psi$, s being measured from the vertex.

   [**Hint.** Parametric equations of this curve are $x = a \cos^3 \theta$, $y = a \sin^3 \theta$. At the vertex (the point of intersection of the given curve with its line of symmetry $y = x$), $\theta = \pi/4$].

   (*ii*) $s = \dfrac{3a}{2} \sin^2 \psi$ when $(a, 0)$ is taken as the fixed point.
8. Find the intrinsic equation of the cardioid $r = a(1 + \cos \theta)$ and show that $s^2 + 9\rho^2 = 16a^2$.
9. (*i*) Show that the intrinsic equation of the equiangular spiral $r = a\, e^{\theta \cot \alpha}$, measured s from the point $(a, 0)$ is $s = a \sec \alpha\, [e^{\psi \cot \alpha} - 1]$.

   (*ii*) Show that intrinsic equation of the curve $r = a\, e^{m\theta}$ is $s = \dfrac{a\sqrt{1+m^2}}{m} [e^{m\psi} - 1]$.
10. Find the intrinsic equation of the curve whose pedal equation is $p = r \sin \alpha$.

Answers

1. $s = c \tan \psi$ **4.** $s = a \log (\tan \psi + \sec \psi)$

5. $s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi)$

6. $s + a\psi = 0$ **8.** $s = 4a \sin \left(\dfrac{\psi}{3} - \dfrac{\pi}{6}\right)$ **10.** $s = e^{\psi \cot \alpha}$.

REVISION EXERCISE

1. Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $a \left(\log_e 2 + \dfrac{15}{16}\right)$.
2. Show that the length of the curve given by $x = a(3 \sin \theta - \sin^3 \theta)$ and $y = a \cos^3 \theta$ measured from $(0, a)$ to any point (x, y) is $\dfrac{3a}{2} (\theta + \sin \theta \cos \theta)$.
3. Prove that the length of the arc of the curve $x = a \sin 2\theta\, (1 + \cos 2\theta)$, $y = a \cos 2\theta\, (1 - \cos 2\theta)$ measured from $(0, 0)$ to (x, y) is $\dfrac{4a}{3} \sin 3\theta$.

Find the intrinsic equation of the following curves : (4 to 8).

4. $x = a\, (2 \cos t - \cos 2t)$, $y = a\, (2 \sin t - \sin 2t)$.
5. $x = a\, (3 \cos \theta - \cos 3\theta)$, $y = a\, (3 \sin \theta - \sin 3\theta)$; s being measured from $\theta = \dfrac{\pi}{2}$.
6. $x = a \sin 2t\, (1 + \cos 2t)$, $y = a \cos 2t\, (1 - \cos 2t)$; the fixed point being the origin.
7. $r = a\theta$, s being measured from the pole.
8. $x = a\, (3 \sin t - \sin^3 t)$, $y = a \cos^3 t$, where s is measured from the point $(0, a)$.

NOTES

9. Show that the length s of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ measured from $(0, a)$ to the point (x, y) is given by $s = \frac{3}{2}\sqrt[3]{ax^2}$.

10. Show that in the parabola $\frac{2a}{r} = 1 + \cos\theta$, $\frac{ds}{d\psi} = \frac{2a}{\sin^3\psi}$.

Hence show that the arc intercepted between the vertex and the extremity of latus rectum is $a\{\sqrt{2} + \log(1+\sqrt{2})\}$.

11. Find the length of any arc of the curve $y^2(a - x) = x^3$ or $r = \frac{a\sin^2\theta}{\cos\theta}$.

Answers

4. $s = 16a\sin^2\frac{\psi}{6}$ **5.** $s = -6a\cos\frac{\psi}{2}$ **6.** $3s = 4a\sin 3\psi$

7. $s = \frac{a'}{2}[\theta\sqrt{\theta^2 + 1} + \log(\theta + \sqrt{\theta^2 + 1})]$ where $\psi = \theta + \tan^{-1}\theta$

8. $4s + 3a(2\psi + \sin 2\psi) = 0$

11. $\frac{a}{2}\left[2\sqrt{\frac{4a - 3x}{a - x}} + \sqrt{3}\cosh^{-1}\frac{6x - 7a}{a}\right]_{x_1}^{x_2}$.

NOTES

6. QUADRATURE (Areas of Curves)

STRUCTURE

ddddd

AREA FORMULAE FOR CARTESIAN EQUATIONS

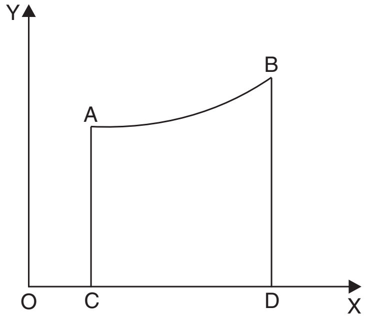
We know that the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b y\,dx$, where $y = f(x)$, is a continuous single valued function and y does not change sign in the interval [a, b].

Thus, if AB is the curve $y = f(x)$ and CA, DB, the ordinates $x = a$, $x = b$, then

$$\text{Area ACDB} = \int_a^b y\,dx.$$

Note 1. We have supposed that y is a strictly increasing function of x in the interval $[a, b]$. But this condition may be removed. If $f(x)$ goes on decreasing as we go from C to D, the above result is still true.

If however, $y = f(x)$ increases in certain parts of the interval. $[a, b]$ and decreases in other parts, as in the adjoining figure, then the area ACDB = the area ACE_1F_1 + the area $F_1E_1E_2F_2$ + ... + the area $E_n\,DBF_n$ ...(1)
where E_1F_1, E_2F_2 ..... $E_n\,F_n$ are the maximum and minimum ordinates of the curve (n being finite), in the interval $[a, b]$.

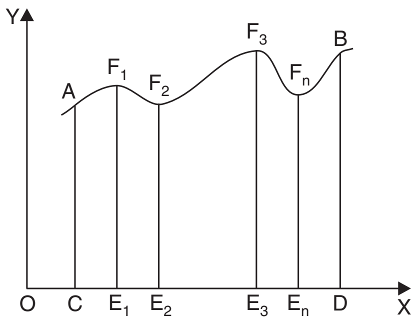


If $c_1, c_2, \ldots c_n$ are the abscissae of the ordinates $E_1F_1, E_2F_2 \ldots, E_nF_n$ respectively, then it follows from (1) that

$$\text{Area ACDB} = \int_a^{c_1} y\,dx + \int_{c_1}^{c_2} y\,dx + \ldots + \int_{c_{n-1}}^{c_n} y\,dx + \int_{c_n}^{b} y\,dx = \int_a^b y\,dx.$$

NOTES

Note 2. The area bounded by the curve AB, the ordinates at A and B, and the x-axis is often called *the area under the curve AB.*



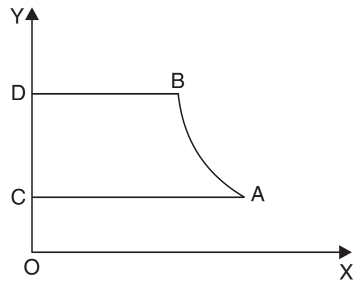
Note 3. *The process of finding the area bounded by a given portion of a curve is called* **quadrature.**

Note 4. Sign of the area. We know that if $y = f(x) > 0$ over the range $a \le x < b$, then the area $\int_a^b f(x)\,dx$ is + ve and if y is – ve in the range $a \le x \le b$, then the area $\int_a^b f(x)\,dx$ is also negative. The curve $y = f(x)$ in this case lies below the x-axis over the range $[a, b]$. Thus, we consider the areas below x-axis as – ve. By the area in such case, we mean the numerical value of the area.

If as x changes from a to b, y = f(x) changes sign at some intermediate point c (say), then the areas from a to c and c to b are, calculated separately and their numerical values are added. Similarly, this result can be extended if y changes sign at more than one intermediate place in the interval $[a, b]$.

TO PROVE THAT THE AREA OF THE CURVE x = f(y) BETWEEN THE y-AXIS AND THE LINES y = c, TO y = d IS GIVEN BY $\int_c^d x\,dy$

The result follows easily from Art. 1, on interchanging x and y with the proper limits of integration.

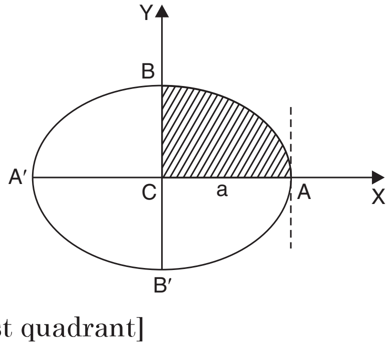


Thus, area ACDB = $\int_c^d x\,dy$, where OC = c, OD = d.

Example 1. *Calculate the area of the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

Sol. The equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ...(*i*)

Since the ellipse is symmetrical about both the axes, therefore the area of the whole ellipse is four times the area under the curve in first quadrant.



i.e., area of ellipse = 4 × area CAB.

Now for area CAB, x varies from 0 to a and from (*i*),

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

[Taking + ve sign before radical sign, in the first quadrant]

∴ (By Art. 1.) Area CAB

$$= \int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\;dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{b}{a}\left|\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1} 1 - 0\right] = \frac{b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4}$$

∴ Whole area of ellipse = 4 × area CAB = 4 × $\pi ab/4 = \pi ab$.

Example 2. *Trace the curve $y^2 (a - x) = x^2(a + x)$ and*

(*a*) *find the area of the loop.*

(*b*) *find the area of the portion bounded by the curve and its asymptote.*

Sol. The equation of the curve is $y^2 (a - x) = x^2 (a + x)$ ...(1)

(*i*) The curve is symmetrical about x-axis, since (1) contains only even powers of y.

(*ii*) The curve passes through the origin and tangents at the origin are given by $y^2 = x^2$ or $y = \pm x$. Since these tangents are real and different, ∴ origin is a node.

If y_c and y_t denote the ordinates of the curve and the tangent $y = x$, for the same value of x, then

$$y_c^{\,2} - y_t^{\,2} = \frac{x^2(a + x)}{a - x} - x^2 = \frac{2x^2}{a - x}$$

which is > 0 for small + ve values of x and is < 0 for $x < 0$. Thus, the curve lies above the tangent $y = x$ in the first quadrant and below the tangent in the second quadrant.

(*iii*) The asymptote parallel to y-axis is given by $a - x = 0$ *i.e.*, $x = a$. The curve has no other asymptote.

(*iv*) The curve meets x-axis, in the point $(-a, 0)$ and the y-axis in $(0, 0)$.

(*v*) From (1), we have $y^2 = \dfrac{x^2(a + x)}{a - x}$

∴ $$2y\frac{dy}{dx} = \frac{(a - x)(2ax + 3x^2) - x^2(a + x)(-1)}{(a - x)^2}$$

$$= \frac{2a^2x + ax^2 - 3x^3 + ax^2 + x^3}{(a - x)^2} = \frac{2x(a^2 + ax - x^2)}{(a - x)^2}$$

or $$\frac{dy}{dx} = \frac{x(a^2 + ax - x^2)}{(a - x)^2}\cdot\frac{\sqrt{a - x}}{x\sqrt{a + x}} = \frac{a^2 + ax - x^2}{(a - x)^{3/2}\sqrt{a + x}}$$

∴ $$\frac{dy}{dx} = 0 \text{ when } x^2 - ax - a^2 = 0$$

or $$x = \frac{a \pm \sqrt{a^2 + 4a^2}}{2} = \frac{a \pm 2.2a}{2} = 1.6a \text{ or } -.6a$$

Thus, the tangent is || to x-axis at $x = -.6a$. The value of $x = 1.6a$ is rejected as we shall see in the next step that y is imaginary when $x = 1.6\,a$.

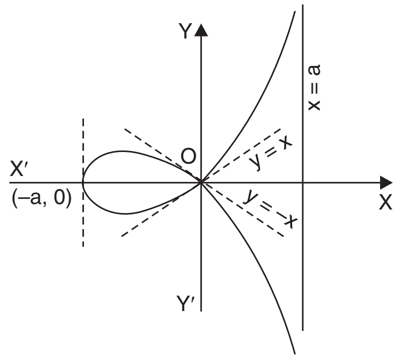
Again, $dy/dx = \infty$ when $x = \pm\, a$. Thus, at $(-a, 0)$, the tangent is parallel to y-axis and as $x \to a$, $y \to \infty$.

∴ $x = a$ is an asymptote, a fact already established.

NOTES

(*vi*) From (1), we have (taking + ve root),

$$y = x\sqrt{\frac{a+x}{a-x}} \qquad \ldots(2)$$



When $x < -a$ or when $x > a$, y is imaginary. Hence no portion of the curve lies beyond the lines $x = \pm a$. Thus, the shape of the curve is as shown in the figure.

Now for the loop, x varies from $-a$ to 0 and the loop is symmetrical about x-axis.

(*a*) ∴ Area of the loop

= 2 × area of the upper half of the loop

$$= 2\int_{-a}^{0} y\,dx = 2\int_{-a}^{0} x\sqrt{\frac{a+x}{a-x}}\,dx \qquad \ldots(3)$$

Now $\int x\sqrt{\frac{a+x}{a-x}}\,dx.$

Rationalising the numerator by multiplying the numerator and denominator by $\sqrt{a+x}$.

$$= \int x\frac{(a+x)}{\sqrt{a^2-x^2}}dx = \int \frac{ax}{\sqrt{a^2-x^2}}dx + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$= a\int x(a^2-x^2)^{-1/2}\,dx + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}dx$$

$$= -\frac{a}{2}\int (a^2-x^2)^{-1/2}(-2x)\,dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}dx - \int \sqrt{a^2-x^2}\,dx$$

$$= -\frac{a}{2}\frac{(a^2-x^2)^{1/2}}{\frac{1}{2}} + a^2\sin^{-1}\frac{x}{a} - \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= -a\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{2} \qquad \ldots(4)$$

∴ From (3), area of the loop

$$= 2\left[-a\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{2}\right]_{-a}^{0}$$

$$= 2\left[-a^2 - \frac{a^2}{2}\sin^{-1}(-1)\right] = 2\left[\frac{\pi a^2}{4} - a^2\right]$$

$$= \frac{2a^2}{4}(\pi - 4) = \frac{a^2}{2}(4-\pi) \text{ numerically} \qquad (\because \pi < 4)$$

(*b*) For half of the area bounded by the curve and its asymptote x varies from 0 to a.

∴ Area between the curve and its asymptote

$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\frac{a+x}{a-x}}\,dx \qquad \text{[Using (4)]}$$

$$= 2\left[-a\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x\sqrt{a^2-x^2}}{2}\right]_0^a$$

$$= 2\left[\frac{a^2}{2}\sin^{-1}1-(-a\,.\,a)\right]=2\left[\frac{a^2}{2}\frac{\pi}{2}+a^2\right]$$

$$= 2a^2\left(\frac{\pi+4}{4}\right)=\frac{a^2}{2}\ (\pi+4).$$

Example 3. *Find the area included between the curve $xy^2 = 4a^2\ (2a - x)$ and its asymptote.*

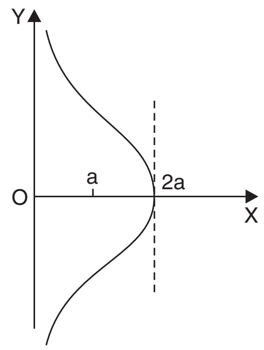
Sol. The equation of the curve is $xy^2 = 4a^2\ (2a - x)$ ...(1)

Let us first trace the curve roughly to find limits of integration for the area.

(*i*) The curve is symmetrical about x-axis.

(*ii*) The curve does not pass through the origin.

(*iii*) The curve meets the x-axis in the point $(2a, 0)$. Shifting the origin to point $(2a, 0)$, equation (1) transforms to $(X + 2a)\ Y^2 = -4aX$, and tangent to the curve at the new origin is $X = 0$, *i.e.*, the new y-axis. Hence at the point $(2a, 0)$ tangent to the curve is || to y-axis.



The curve does not meet the y-axis.

(*iv*) $x = 0$ *i.e.*, y-axis is the only asymptote of the curve.

(*v*) From (1), $$y = \pm\, 2a\sqrt{\frac{2a-x}{x}}$$ ...(2)

When x is negative and $x > 2a$, y is imaginary. Hence no portion of curve lies to the left of y-axis and to right of line $x = 2a$.

From (1), $$x = \frac{8a^3}{y^2+4a^2}\ .$$

Now area required

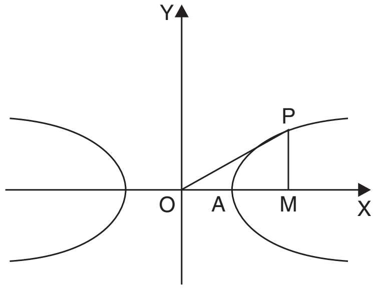
= 2 × area between the upper half of the curve, y-axis and x-axis

$$= 2\int_0^\infty x\,dy = 2\int_0^\infty \frac{8a^3}{y^2+4a^2}\,dy$$

$$= 16a^3\ .\ \frac{1}{2a}\ .\left|\tan^{-1}\frac{y}{2a}\right|_0^\infty$$

$$= 8a^2\left(\frac{\pi}{2}-0\right) = 4\pi a^2.$$

Example 4. *If A is the vertex, O the centre and P any point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, prove that $x = a\cosh\dfrac{2S}{ab}$, $y = b\sinh\dfrac{2S}{ab}$, where S is the sectorial area AOP.*



Sol. Draw PM ⊥ on OX.

S = Area OAP

= Area of ΔOMP – area AMP

$$= \tfrac{1}{2}xy - \int_a^x y\,dx \text{ where}$$

NOTES

$$y = \frac{b}{a}\sqrt{x^2 - a^2} \quad \left(\because \text{ From the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, y = \frac{b}{a}\sqrt{x^2 - a^2}\right)$$

$$= \frac{x}{2}\cdot\frac{b}{a}\sqrt{x^2 - a^2} - \frac{b}{a}\int_a^x \sqrt{x^2 - a^2}\; dx$$

$$= \frac{bx\sqrt{x^2 - a^2}}{2a} - \frac{b}{a}\left[\frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right]_a^x$$

or
$$S = \frac{bx\sqrt{x^2 - a^2}}{2a} - \frac{b}{a}\left[\frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right] \qquad | \because \quad \cosh^{-1} 1 = 0$$

or
$$S = \frac{ab}{2}\cosh^{-1}\frac{x}{a}$$

$$\therefore \quad \frac{2S}{ab} = \cosh^{-1}\frac{x}{a} \quad \therefore \quad x = a\cosh\frac{2S}{ab}$$

Hence
$$y = \frac{b}{a}\sqrt{x^2 - a^2} = \frac{b}{a}\sqrt{a^2\cosh^2\frac{2S}{ab} - a^2}$$

or
$$y = b\sinh\frac{2S}{ab}.$$

EXERCISE 6.1

1. Find the area bounded by each of the following curves, the x-axis and the ordinates $x = 0$, $x = h$:

(*i*) $y = e^x$ (*ii*) $y = c\cosh\dfrac{x}{c}$.

2. (*a*) Find the areas bounded by each of the following curves, the x-axis and the ordinates $x = a$, $x = b$:

(*i*) $xy = c^2$ (*ii*) $y = \log x$.

$$\left[\textbf{Hint.} \int \log x\, dx = \int \underset{\text{I}}{\log x}\,.\,\underset{\text{II}}{1}\, dx = (\log x)\, x - \int \frac{1}{x} x\, dx = x\log x - x = x(\log x - 1).\right]$$

(*b*) In the catenary $y = a\cosh\dfrac{x}{a}$, prove that the area between the curve, the x-axis and the ordinates of two points on the curve varies as the length of the intervening arc.

3. Show that the area cut off a parabola by any double ordinate is two-thirds of the corresponding rectangle contained by the double ordinate and its distance from the vertex.

4. Obtain the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

5. (*a*) Trace the curve $a^2y^2 = a^2x^2 - x^4$. Find the whole area within it.

(*b*) Find the area enclosed by the curve.

$$y^2 = x^2 - x^4.$$

6. Trace the curve $ay^2 = x^2(a - x)$ and show that the area of its loop is $\dfrac{8}{15}a^2$.

$$\left[\textbf{Hint.} \int x\sqrt{a - x}\; dx = \int [a - (a - x)]\sqrt{a - x}\, dx = a\int (a - x)^{1/2} dx - \int (a - x)^{3/2} dx.\right]$$

7. (*a*) Find the area of a loop of the curve $a^3y^2 = x^4(b + x)$.

$$\left[\textbf{Hint.} \text{ To evaluate } \int x^2\sqrt{b + x}\, dx, \text{ put } \sqrt{b + x} = t.\right]$$

(*b*) Find the area of a loop of the curve $3ay^2 = x(x - a)^2$

(*c*) Show that the area of a loop of the curve $y^2 = x(4 - x^2)$ is $\frac{16}{3}$.

8. Trace the curve $a^4y^2 = x^5(2a - x)$ and prove that its area is to that of the circle whose radius is a as 5 to 4.

9. Prove that the area of the curve $a^2x^2 = y^3(2a - y)$ is equal to that of the circle whose radius is a.

[**Hint.** Area = $\int x dy$.]

10. Find the area enclosed by the curve $xy^2 = 4(2 - x)$ and y-axis.

11. Find the area included between the curve $x^2y = 4a^2(2a - y)$ and its asymptote.

12. (*i*) Show that the area of the loop of the curve

$$ay^2 = (x - a)(x - 5a)^2 \text{ is } \frac{256}{15} a^2.$$

(*ii*) Show that the area of the loop of the curve $y^2 = x^3 - 11x^2 + 35x - 25$ is $\frac{256}{15}$.

[**Hint.** Equation of the curve is $y^2 = x^3 - 11x^2 + 35x - 25$.

$$\therefore \quad y = x^3 - x^2 - 10x^2 + 10x + 25x - 25$$
$$= x^2(x - 1) - 10x(x - 1) + 25(x - 1)$$
$$= (x - 1)(x^2 - 10x + 25)$$
$$= (x - 1)(x - 5)^2.$$

It is part (*i*) with $a = 1$.]

13. (*a*) Find the whole area of the curve $y^2 = x^2\left(\frac{a^2 - x^2}{a^2 + x^2}\right)$.

$$\left[\textbf{Hint. It will be found that area} = 4\int_0^a x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\,dx. \text{ Put } \sqrt{a^2 + x^2} = t.\right]$$

(*b*) Find the whole area of the curve $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.

[**Hint.** It is same as part (*i*).]

14. Find the area of a loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$. Also find the area between this curve and its asymptote.

15. Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptote.

16. Find the area between the curve $y^2 = \frac{x^3}{2a - x}$ and its asymptote.

17. Show that the area of the infinite region enclosed between the curve $x^2(1 - y)y = 1$ and its asymptote is 2π.

18. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $x = a\cos\frac{2S}{ab}$, $y = b\sin\frac{2S}{ab}$, where S is the sectorial area bounded by the ellipse, x-axis and the line joining (0, 0) to (x, y).

Answers

1. (*i*) $e^h - 1$ (*ii*) $c^2 \sinh\frac{h}{c}$

2. (*a*) (*i*) $c^2 \log\frac{b}{a}$ (*ii*) $b\log b - a\log a - (b - a)$ **4.** $\frac{8}{3}a^2$

5. (*a*) $\frac{4}{3}a^2$ (*b*) $\frac{4}{3}$

NOTES

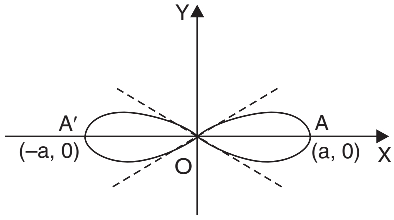
7. (*a*) $\frac{32}{105}\, b^{7/2}\, a^{-3/2}$ (*b*) $\frac{8a^2}{5\sqrt{3}}$

10. 4π 11. $4\pi a^2$

13. (*a*) $a^2(\pi - 2)$ (*b*) $a^2(\pi - 2)$

14. $\frac{a^2}{2}(4 - \pi), \frac{a^2}{2}(\pi + 4)$ 15. $4a^2$

16. $3\pi a^2$.

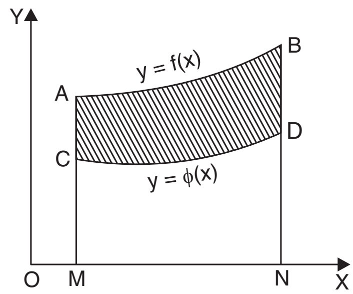


AREA BETWEEN TWO CURVES

To prove that the area bounded by the curves $y = f(x)$, $y = \phi(x)$ and the ordinates $x = a$, $x = b$ is $\int_a^b (y_1 - y_2)\, dx$

where y_1 is the 'y' of the upper curve and y_2 that of the lower curve.

Let AB, CD be the curves $y = f(x)$, $y = \phi(x)$ and MCA, NDB be the ordinates $x = a$, $x = b$. Then area ACDB = area AMNB – area CMND



$$= \int_a^b f(x)\, dx - \int_a^b \phi(x)\, dx$$

$$= \int_a^b [f(x) - \phi(x)]\, dx$$

$$= \int_a^b (y_1 - y_2)\, dx$$

where $y_1 = f(x)$ is the 'y' of the upper curve AB and $y_2 = \phi(x)$, is the 'y' of lower curve CD.

Cor. The area bounded by the curves $x = f(y)$, $x = F(y)$ and the abscissae $y = c$, $y = d$ is given by

$$\int_c^d [(x \text{ of outer curve}) - (x \text{ of inner curve })]\, dy.$$

Caution. *While applying formula of Art. 3 one must see the figure of the problem very carefully. In some problems we may have to use $\int (y_1 + y_2)\, dx$ (see example 2).*

Example 1. *Find the area included between the parabola $x^2 = 4ay$ and the curve*

$$y(x^2 + 4a^2) = 8a^3.$$

Sol. The equation of the parabola is $x^2 = 4ay$ ...(1)

It is an upward parabola with y-axis as its axis and the origin as its vertex.

The equation of the curve is $y = \frac{8a^3}{x^2 + 4a^2}$ ...(2)

(*i*) The curve (2) is symmetrical about y-axis.

(*ii*) The curve does not pass through the origin. The curve does not meet x-axis and it meets y-axis in the point $(0, 2a)$.

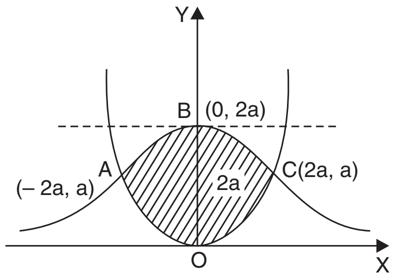
(*iii*) $y = 0$ is an asymptote of the curve (2).

From (2), $y > 0$.

$\therefore$ The curve given by equation (2) lies above the axis of x.

Thus, the shape of the curve (2) is as shown in the figure (along with the parabola (1)).

NOTES



To find abscissae of points of intersection eliminating y between (1) and (2), [*i.e.*, putting the value of y from (2) in (1)], we get

$$x^2 = 4a \cdot \frac{8a^3}{x^2 + 4a^2}$$

or $$x^4 + 4a^2x^2 - 32a^4 = 0$$

or $$(x^2 + 8a^2)(x^2 - 4a^2) = 0$$

Rejecting $x^2 + 8a^2 = 0$ which gives imaginary values of x, we have $x^2 - 4a^2 = 0$ or $x = \pm 2a$.

When $x = \pm 2a$, $y = a$ [from (1)].

Hence the points of intersection are A(– $2a$, a), C($2a$, a).

∴ Required area OABC = 2 × area OBC

$$= 2\int_0^{2a} (y_1 - y_2)\, dx = 2\int_0^{2a} \left(\frac{8a^3}{x^2 + 4a^2} - \frac{x^2}{4a}\right) dx$$

$$= 2\left| 8a^3 \cdot \frac{1}{2a} \tan^{-1} \frac{x}{2a} - \frac{1}{4a} \cdot \frac{x^3}{3} \right|_0^{2a}$$

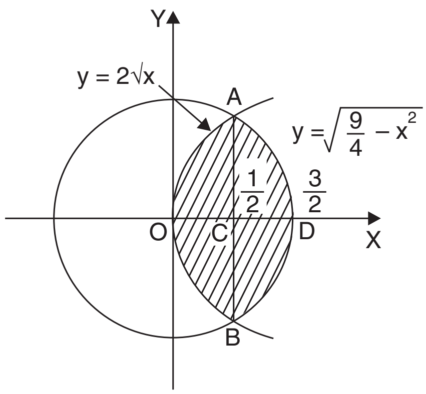
$$= 2\left[4a^2(\pi/4 - 0) - \frac{1}{12a}(8a^3 - 0)\right]$$

$$= 2\left(\pi a^2 - \frac{2}{3}a^2\right) = \frac{2}{3}a^2(3\pi - 2).$$

Remark. We know that **the equation of the circle whose centre is (α, β) and radius a is $(x - \alpha)^2 + (y - \beta)^2 = a^2$.**

If centre (α, β) is (0, 0) ; then equation of circle is $x^2 + y^2 = a^2$.

Example 2. *Find the area common to the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.*



Sol. The required area is the area common to the interiors of the parabola

$$y^2 = 4x \qquad \text{...(1)}$$

[Parabola (1) is a rightward parabola and is symmetrical about x-axis.]

and the circle $$4x^2 + 4y^2 = 9 \qquad \text{...(2)}$$

Dividing every term of eqn. (2) by 4,

$$x^2 + y^2 = \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

which is a circle whose centre is origin and radius is $\frac{3}{2}$.

Putting $y^2 = 4x$ from (1) in (2), $4x^2 + 16x - 9 = 0$

∴ $$x = \frac{-16 \pm \sqrt{256 + 144}}{8} = \frac{-16 \pm 20}{8} = \frac{1}{2}, -\frac{9}{2}.$$

When $x = -\frac{9}{2}$, from (1), $y^2 = -18$ is negative and hence y is imaginary and hence impossible.

When $x = \frac{1}{2}$, from (1), $y^2 = 4x = 4 \times \frac{1}{2} = 2$

$\therefore$ The two points of intersection of parabola (1) and circle (2) are

$$A\left(\frac{1}{2}, \sqrt{2}\right) \text{ and } B\left(\frac{1}{2}, -\sqrt{2}\right).$$

Both the curves are symmetrical about x-axis.

For the parabola (1), $y = 2\sqrt{x}$ in the first quadrant.

For the circle (2), $4y^2 = 9 - 4x^2$ or $y^2 = \frac{9}{4} - x^2$

or $$y = \sqrt{\frac{9}{4} - x^2} \text{ in first quadrant.}$$

Required area OADBO (shaded)

$$= 2 \times \text{Area OADO} = 2\,[\text{Area OAC} + \text{Area CAD}]$$

$$= 2\left[{}^{*}\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\,dx\right] \qquad [\text{Area} = \textstyle\int y\,dx]$$

$$= 2\left[\left\{2 \cdot \frac{x^{3/2}}{3/2}\right\}_0^{1/2} + \left\{\frac{x\sqrt{\frac{9}{4} - x^2}}{2} + \frac{\frac{9}{4}}{2}\sin^{-1}\frac{x}{3/2}\right\}_{1/2}^{3/2}\right]$$

$$= 2\left[\frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{9}{8}\sin^{-1} 1 - \frac{\frac{1}{2}\sqrt{2}}{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{\sqrt{2}}{3} + \frac{9}{8} \cdot \frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right] = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} + \frac{\sqrt{2}}{6}.$$

Example 3. *Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.*

Sol. The equations of the two circles are $x^2 + y^2 = 1$ ...(1)

and $$(x - 1)^2 + y^2 = 1 \qquad ...(2)$$

The first circle has centre at the origin and radius 1. The second circle has centre at (1, 0) and radius 1. Both are symmetrical about the x-axis.

For points of intersections of circles (1) and (2)

From (1), $y^2 = 1 - x^2$

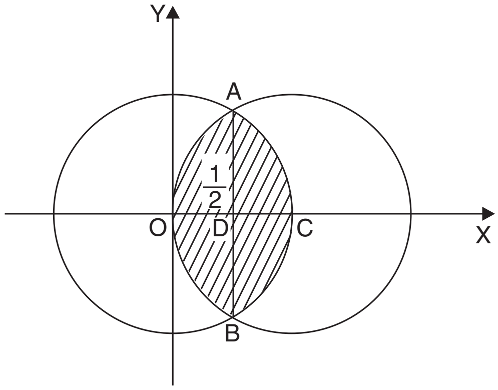
Putting $y^2 = 1 - x^2$ in eqn. (2), $(x - 1)^2 + 1 - x^2 = 1$

or $$x^2 + 1 - 2x + 1 - x^2 = 1 \quad \text{or} \quad -2x + 1 = 0 \quad \therefore \quad x = \tfrac{1}{2}$$

---

*Limits of integration for parabola are $x = 0$ to x of point of intersection and for circle are x of point of intersection to x = radius of circle.

Also in the first quadrant,



From (1), $y = \sqrt{1 - x^2}$

From (2), $y = \sqrt{1 - (x - 1)^2}$.

Required area OACBO (shaded)

= 2 × Area OAC

= 2 [Area OAD + Area DAC]

$$= 2\left[\int_0^{1/2} y \text{ of circle (2) } dx + \int_{1/2}^{1} y \text{ of circle (1) } dx\right]$$

(Note)

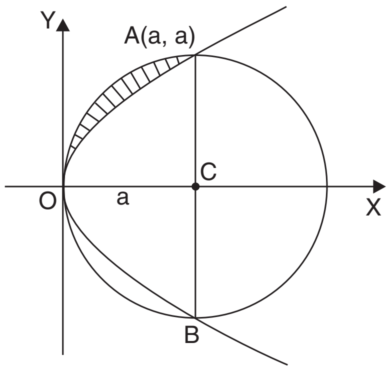
$$= 2\left[\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1}\sqrt{1-x^2}\,dx\right]$$

$$= 2\left[\left\{(\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}(x-1)\right\}_0^{1/2} + \left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right\}_{1/2}^{1}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\left(-\frac{1}{2}\right)\right\} - \{\sin^{-1}(-1)\} + \sin^{-1}1 - \left\{\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

Example 4. *Find the area of the region lying above x-axis and included between the circle $x^2 + y^2 = 2ax$ and parabola $y^2 = ax$.*



Sol. Equation of circle is $x^2 + y^2 = 2ax$ ...(1)

Equation of parabola is $y^2 = ax$ ...(2)

Adding a^2 to both sides of eqn. (1),

$$x^2 - 2ax + a^2 + y^2 = a^2$$

or $$(x - a)^2 + y^2 = a^2$$

$$\therefore \quad y = \sqrt{a^2 - (x-a)^2} \quad ...(3)$$

which represents a circle whose centre is $(a, 0)$ and radius is a. Also circle (1) passes through the origin.

NOTES

To find points of intersections of circle (1) and parabola (2).

Putting $y^2 = ax$ from (2) in (1), we have

$x^2 + ax = 2ax$ or $x^2 - ax = 0$ or $x(x - a) = 0$ $\therefore$ $x = 0, a$

When $x = 0$, $y = 0$; when $x = a$, $y^2 = a^2$ so that $y = a$ (in 1st quadrant)

$\therefore$ The two points of intersection are O(0, 0) and A(a, a).

Required area (shaded)

$$= \int_0^a (y \text{ of circle (1) } i.e., (3))\, dx - \int_0^a (y \text{ of parabola (2)})\, dx$$

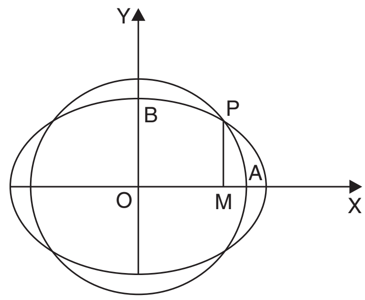
$$= \int_0^a \sqrt{a^2 - (x-a)^2}\, dx - \int_0^a \sqrt{a}\sqrt{x}\, dx$$

$$= \left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}\right]_0^a - \sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a$$

$$= (0 + 0) - \left[0 + \frac{a^2}{2}\sin^{-1}(-1)\right] - \frac{2}{3}\sqrt{a}\ a^{3/2}$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{2a^2}{3} = \frac{a^2}{12}(3\pi - 8).$$

Example 5. *Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.*



Sol. Let P be the point of intersection in the first quadrant, of the circle and the ellipse, as shown in the figure.

Draw MP$\perp$ x-axis.

To find abscissae of the points of intersection, eliminating y^2 from the two equations of the curves, we have (Putting $y^2 = 4 - x^2$ from first equation in the second equation)

$$x^2 + 4(4 - x^2) = 9 \quad \text{or} \quad 3x^2 = 7$$

which gives $x = \sqrt{7/3}$ for P.

[Rejecting – ve value as P lies in first quadrant]

Now $y = \frac{1}{2}\sqrt{9 - x^2}$, for the ellipse and $y = \sqrt{4 - x^2}$, for the circle.

Since the circle and ellipse are symmetrical about the axes.

$\therefore$ Required area common to circle and ellipse

= 4 × area OAPB

= 4 [area OBPM + area PMA]

$$= 4\left[\int_0^{\sqrt{7/3}} \frac{1}{2}\sqrt{9 - x^2}\, dx + \int_{\sqrt{7/3}}^2 \sqrt{4 - x^2}\, dx\right]$$

$$= 2\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^{\sqrt{7/3}} + 4\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{7/3}}^2$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[\sqrt{\frac{7}{3}} \cdot \sqrt{9 - \frac{7}{3}} + 9 \sin^{-1} \sqrt{\frac{7}{27}} - 0\right] + 2\left[0 + 4 \sin^{-1} 1 - \sqrt{\frac{7}{3}}\sqrt{4 - \frac{7}{3}} - 4 \,.\, \sin^{-1} \sqrt{\frac{7}{12}}\right]$$

$$= \sqrt{\frac{7}{3}}\left[\sqrt{\frac{20}{3}} - 2\sqrt{\frac{5}{3}}\right] + 9 \sin^{-1} \sqrt{\frac{7}{27}} + \frac{8\pi}{2} - 8 \sin^{-1} \sqrt{\frac{7}{12}}$$

$$= 4\pi + 9 \sin^{-1} \sqrt{\frac{7}{27}} - 8 \sin^{-1} \sqrt{\frac{7}{12}} .$$

EXERCISE 6.2

1. (*a*) Find the area bounded by the parabola $x^2 = 8y$ and the line $x - 2y + 8 = 0$.

(*b*) Show that the area of the segment cut off from the parabola $y^2 = 2x$ by the straight line

$$y = 4x - 1 \text{ is } 9/32.$$

2. (*a*) Find the area of the segment cut off from the parabola $y^2 = 4x$ by the straight line

$$2x - 3y + 4 = 0.$$

(*b*) Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$.

3. Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y = - x$.

4. (*a*) Find the area common to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

(*b*) Find the area included between curves $y^2 = 4bx$ and $x^2 = 4ay$.

(*c*) Find the area enclosed between the parabola $y^2 = 8x$ and $x^2 = 12y$.

5. Find the area common to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$.

6. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\frac{16}{3} a^2(8\pi - \sqrt{3})$.

7. Find the area of the smaller portion enclosed by the curves $y^2 = 8x$ and $x^2 + y^2 = 9$.

8. Show that the area enclosed between the parabolas

$$y^2 = 4a(x + a) \text{ and } y^2 = - 4a(x - a) \text{ is } \frac{16a^2}{3} .$$

9. Find the area enclosed between the parabolas

$$y^2 = 4a\,(x + a) \text{ and } y^2 = 4b(b - x).$$

[**Hint.** The points of intersection of the two parabolas are

$$x = b - a,\ y = \pm 2\sqrt{ab} .$$

$$\text{Required Area} = 2\left[\int_{-a}^{b-a} \sqrt{4a\,(x + a)}\,dx + \int_{b-a}^{b} \sqrt{4b(b - x)}\,dx\right].$$

10. Find the area of the region lying above x-axis and included between the circle $x^2 + y^2 = 4x$ and the parabola $y^2 = 2x$.

[It is Example 4 with $a = 2$]

NOTES

Answers

1. (*a*) 36 Numerically **2.** (*a*) $\frac{1}{3}$ (*b*) $\frac{16}{5}$

4. (*a*) $\frac{16}{3}a^2$ (*b*) $\frac{16ab}{3}$ (*c*) 32

5. $a^3\left(3\sqrt{3}+\frac{4\pi}{3}\right)$ **7.** $\frac{2\sqrt{2}}{3}+\frac{9\pi}{2}-9\sin^{-1}\frac{1}{3}$

9. $\frac{8}{3}(a+b)\sqrt{ab}$ **10.** $\frac{1}{3}(3\pi-8)$.

AREA FORMULAE FOR CURVES GIVEN BY PARAMETRIC EQUATIONS

(*i*) *To prove that the area bounded by the curves x = f(t), y = φ(t), the x-axis and the ordinates at the points where, t = a, t = b is given by*

$$\int_a^b y\frac{dx}{dt}\,dt \qquad ...(1)$$

The parametric equations of the curve are $x = f(t)$, $y = \phi(t)$.

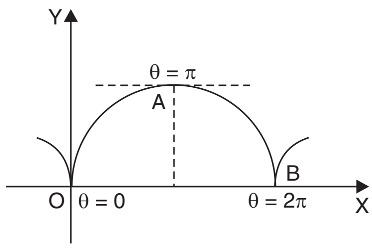
From Art. 1, we know that

Required area = $\int_{t=a}^{t=b} y\,dx = \int_a^b y\frac{dx}{dt}\,dt$.

(*ii*) *Similarly, the area bounded by the curve x = f(t), y = φ(t), the y-axis and the abscissae at the points where t = c, t = d is given by*

$$\int_c^d x\frac{dy}{dt}\,dt \qquad ...(2)$$

Example 1. *Find the area included between the cycloid x = a(θ – sin θ), y = a(1 – cos θ) and its base.*



Sol. For the first half of the cycloid, θ varies from 0 to π.

∴ Required area bounded by the cycloid.

$$= 2\int_0^\pi y\,.\,\frac{dx}{d\theta}\,d\theta \qquad \text{[By Art. 4 (i)]}$$

$$= 2\int_0^\pi a(1-\cos\theta)\,.\,a(1-\cos\theta)\,d\theta$$

[∵ $y = a(1-\cos\theta)$ and $x = a(\theta-\sin\theta)$]

$$= 2a^2\int_0^\pi (1-\cos\theta)^2\,d\theta = 2a^2\int_0^\pi\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 2a^2\int_0^\pi 4\sin^4\frac{\theta}{2}\,d\theta$$

Put $\frac{\theta}{2} = t$ ∴ $\theta = 2t$ ∴ $d\theta = 2dt$

When $\theta = 0$, $t = 0$; when $\theta = \pi$, $t = \frac{\pi}{2}$

$$= 8a^2\int_0^{\pi/2}\sin^4 t\,.\,2dt = 16a^2\int_0^{\pi/2}\sin^4 t\,dt = 16a^2\,.\,\frac{3.1}{4.2}\,.\,\frac{\pi}{2} = 3\pi a^2.$$

EXERCISE 6.3

1. (*a*) Find the area of the curve $x = a \cos^3 t$, $y = b \sin^3 t$ or the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

$$\left[\textbf{Hint. } \text{Area of curve} = 4\int_0^{\pi/2} y\frac{dx}{dt}\,.\,dt = 4\int_0^{\pi/2} b \sin^3 t\,(-\,3a\cos^2 t \sin t)\,dt\right]$$

(*b*) Find the area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$

or $x = a \cos^3 t$, $y = a \sin^3 t$.

(*c*) Show that the area of the ellipse $x = a \cos t$, $y = b \sin t$ is πab.

2. Prove that the whole area between the four infinite branches of the tractrix.

$$x = a \cos t + \frac{a}{2} \log \tan^2 \frac{t}{2}\ .\ y = a \sin t \text{ is } \pi a^2.$$

$$\left[\textbf{Hint. } \text{Area} = 4\int_0^{\pi/2} y\frac{dx}{dt} dt.\right]$$

3. Find the whole area of the curve $x = a\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2at}{1+t^2}$.

[**Hint.** Eliminating t (by squaring and adding), we have $x^2 + y^2 = a^2$ which is a circle of radius a. Hence area $= \pi a^2$.]

4. Find the area included between the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ and its base.

5. Find the area of the loop of the curve $x = a(1 - t^2)$, $y = at(1 - t^2)$.

[**Hint.** Dividing $t = \frac{y}{x}$, putting this value of t in $x = a(1 - t^2)$ we have $ay^2 = x^2(a - x)$

$\therefore$ Area of loop $= 2\int_0^a y\,dx$. Also See Q. 6, Exercise 6 (*a*).]

6. Show that the area bounded by the cissoid $x = a \sin^2 t$, $y = a\frac{\sin^3 t}{\cos t}$ and its asymptote is $\frac{3\pi a^2}{4}$.

[**Hint.** Eliminating t, the equation of the curve is $y^2(a - x) = x^3$.]

7. Show that the area of a loop of the curve $x = a \sin 2t$, $y = a \sin t$ is $\frac{4}{3}a^2$.

[**Hint.** Change to cartesians for tracing.]

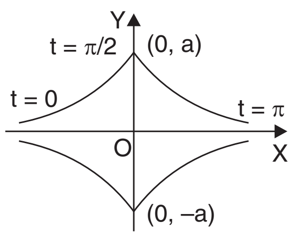
Answers

1. (*a*) $\frac{3\pi}{8} ab$ (Numerically) (*b*) $\frac{3}{8}\pi a^2$

3. πa^2

4. $3\pi a^2$ **5.** $\frac{8}{15}a^2$ (Numerically).

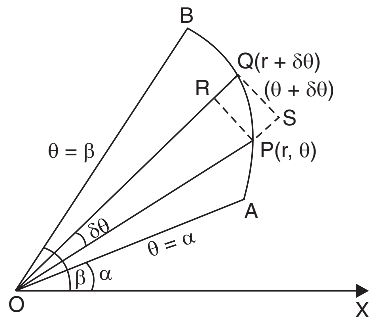
Note. For sake of convenience of the reader, the shape of the curve in question (2) is being given here.



AREA FORMULA FOR CURVES GIVEN BY POLAR EQUATIONS

NOTES

To prove that the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is $\int_\alpha^\beta \frac{1}{2} \mathbf{r^2}\, \mathbf{d}\theta$, *where $r = f(\theta)$ is a continuous and single valued function of θ in* $[\alpha, \beta]$.



Let AB be the curve $r = f(\theta)$, and OA, OB the radii vectors

$$\theta = \alpha,\ \theta = \beta.$$

Let P(r, θ) be *any point* on the curve. Let the *sectorial area* OAP be denoted by A, which clearly is a function of θ.

Let Q $(r + \delta r, \theta + \delta\theta)$ be a point on the curve close to P.

Then the area OPQ = δA.

With O as centre and OP, OQ as radii, draw circular arcs to meet OQ, OP (produced) in R and S respectively.

Then the area OPQ lies between areas of two circular sectors OPR and OSQ.

i.e., δA lies between $\frac{1}{2} r^2 \delta\theta$ and $\frac{1}{2} (r + \delta r)^2 \delta\theta$.

[$\because$ area of a circular sector = $\frac{1}{2}$ (radius)^2 × circular measure of angle of the sector]

$\therefore$ $\dfrac{\delta A}{\delta\theta}$ lies between $\frac{1}{2} r^2$ and $\frac{1}{2} (r + \delta r)^2$

Proceeding to limits as $\delta\theta \to 0$ and $\therefore$ $\delta r \to 0$, we get

$\dfrac{dA}{d\theta}$ lies between $\frac{1}{2} r^2$ and a quantity which $\to \frac{1}{2} r^2$ $\quad\therefore\quad \dfrac{dA}{d\theta} = \frac{1}{2} r^2$.

Integrating both sides within the limits α and β.

$$\therefore \qquad \int_\alpha^\beta \tfrac{1}{2} r^2\, d\theta = \int_\alpha^\beta \frac{dA}{d\theta} d\theta = [A]_\alpha^\beta$$

$$= (\text{value of A when } \theta = \beta) - (\text{value of A when } \theta = \alpha)$$

$$= \text{area OAB} - 0$$

Hence area **OAB** = $\displaystyle\int_\alpha^\beta \frac{\mathbf{1}}{\mathbf{2}} \mathbf{r^2}\, \mathbf{d}\theta$.

Note 1. In the above result, we have supposed that r is an increasing function of θ in the interval $[\alpha, \beta]$. The result is still true even if the radius vector r decreases as θ varies from α to β.

Note 2. In some cases, it is more convenient to transform a given cartesian equation into polars than to solve for y. In such cases, the formula of the present article is applied after changing to polars.

Example 1. *Find the area of a loop of the curve $r = a \cos 2\theta$ and hence find the total area of the curve.*

Sol. The curve $r = a \cos 2\theta$ has **four** equal loops (see Note 1, the end of this example).

For a loop putting $r = 0$, we get

$$\cos 2\theta = 0 \quad \therefore \quad 2\theta = \frac{-\pi}{2}, \frac{\pi}{2} \qquad \text{(two consecutive values)}$$

(See Note 2 at the end of this example)

$$\therefore \qquad \theta = \frac{-\pi}{4}, \frac{\pi}{4}$$

i.e., for the first loop of the curve θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$

$\therefore$ Area of one loop of the curve

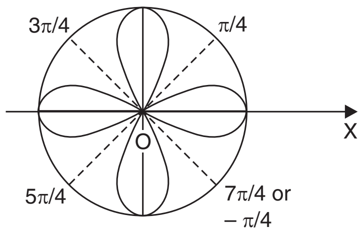
$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2 \cos^2 2\theta \, d\theta$$

$$= \frac{a^2}{2} \times 2 \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$\left[\because \cos^2 2\theta \text{ is an even function of } \theta \text{ and for an even function } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\right]$

$$= a^2 \int_0^{\pi/2} \cos^2 t \frac{dt}{2} \qquad \text{(Putting } 2\theta = t\text{)}$$

$$\frac{a^2}{2}\frac{1}{2}\frac{\pi}{2} = \frac{\pi a^2}{8}$$



$\therefore$ Total area of the curve

$= 4 \times$ (Area of one loop)

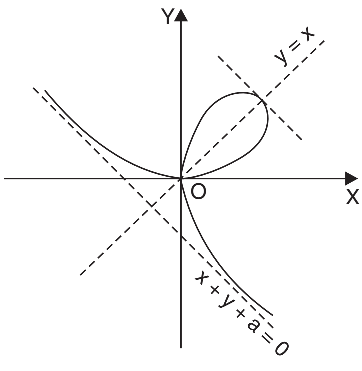
$$= 4 \times \frac{\pi a^2}{8} = \frac{\pi a^2}{2}.$$

Note. 1. It should be carefully remembered that the curves $r = a \sin n\theta$ or $r = a \cos n\theta$ have *n equal loops if n is odd* and $2n$ *equal loops if n is even.*

Note. 2. To find the limits of integration for a loop we generally put $r = 0$ and find *two consecutive* values of θ.

Example 2. *Find the area of the loop of the folium of Descrates*, $x^3 + y^3 = 3axy$.

Sol. The equation of the curve is $x^3 + y^3 = 3axy$



Changing to polars, (see Note 2, Art. 5) by putting $x = r\cos\theta$, $y = r\sin\theta$; (*i*) reduces to

$$r^3 (\sin^3\theta + \cos^3\theta) = 3ar^2 \sin\theta\cos\theta$$

or
$$r = \frac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}$$

For the loop, putting $r = 0$, we get $\sin\theta\cos\theta = 0$ *i.e.*, $\sin\theta = 0$ and $\cos\theta = 0$ or $\theta = 0, \frac{\pi}{2}$.

Hence for the loop θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Area of the loop

$$= \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \frac{9a^2}{2} \int_0^{\pi/2} \frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta + \cos^3\theta)^2} d\theta$$

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{(1+\tan^3\theta)^2}\, d\theta \text{ (On dividing the num. and den. by } \cos^6\theta)$$

(Putting $\tan^3\theta = t$, so that $3\tan^2\theta \sec^2\theta\, d\theta = dt$)

NOTES

$$= \frac{3a^2}{2}\int_0^{\infty} \frac{dt}{(1+t)^2}$$

$$= \frac{3a^2}{2}\left[-\frac{1}{1+t}\right]_0^{\infty} = \frac{3a^2}{2}\,[0-(-1)] = \frac{3a^2}{2}.$$

EXERCISE 6.4

1. Find the area of any sector of each of the following curves :
   (*i*) $r\theta = a$ ($\theta = \alpha$ to $\theta = \beta$)
   (*ii*) $r = a\, e^{\theta \cot \alpha}$ ($\theta = \beta$ to $\theta = \beta + \gamma$, (γ being $< 2\pi$)).
2. Find the area of one loop of the following curves :
   (*i*) $r = a \sin 2\theta$ (*ii*) $r = a \sin 3\theta$
   (*iii*) $r = a \sin 4\theta$ (*iv*) $r = a \sin n\theta$. Also state the total area.
3. Show that the area of a loop of $r = a \cos n\theta$ is $\frac{\pi a^2}{4n}$. Also prove that the whole area is $\frac{\pi a^2}{4}$ or $\frac{\pi a^2}{2}$ according as n is odd or even.
4. Show that the area contained between the circle $r = a$ and the curve $r = a \cos 5\theta$ is equal to three-fourths of the area of the circle.
5. Find the area of a loop of the curve $r = a\theta \cos\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.
6. Trace the curve $r^2 = a^2 \cos 2\theta$ and find its area.
7. Find the areas bounded by :
   (*i*) the cardioid $r = a(1 - \cos\theta)$
   (*ii*) the cardioid $r = a(1 + \cos\theta)$.
8. Show that the whole area of the curve represented by the equation
   $r = a + b\cos\theta$, assuming $a > b$ is $\frac{\pi}{2}(2a^2 + b^2)$.
9. Prove that the sum of the area of the two loops of the limacon
   $r = a + b\cos\theta$ ($a < b$) is equal to $\frac{\pi}{2}(2a^2 + b^2)$.
10. Find the area bounded by $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.
    [**Hint.** The curve is symmetrical about the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$.
    $\therefore$ Area $= 4\int_0^{\pi/2} \frac{1}{2} r^2\, d\theta.$]
11. Show that the area of a loop of the curve $r = \sqrt{3}\cos 3\theta + \sin 3\theta$ is $\frac{\pi}{3}$.
12. Find the area of the loop of the following curves :
    (*a*) $(x^2 + y^2)^2 = 4axy^2$ (*b*) $x^4 + y^4 = 4a^2 xy$
    (*c*) $x^5 + y^5 = 5ax^2y^2$.

13. Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of a loop of the curve $r^2 = a^2 \cos 2\theta$.

14. Show that the area bounded by the curve $p = f(r)$ and the two radii vectors $r = a$, $r = b$ is $\dfrac{1}{2}\displaystyle\int_a^b \sqrt{\dfrac{pr\,dr}{r^2 - p^2}}$.

$\left[\textbf{Hint.}\ \text{Area} = \displaystyle\int \frac{1}{2} r^2\, d\theta = \int \frac{1}{2} r^2 \frac{d\theta}{dr}\, dr = \int \frac{1}{2} r \tan\phi\, dr \left(\because\ \tan\phi = r\frac{d\theta}{dr}\right).\right.$

$\left.\text{Now use } p = r \sin\phi.\right]$

Answers

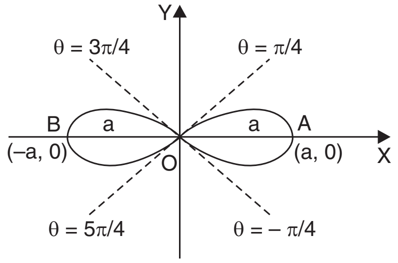
1. (*i*) $a^2\left(\dfrac{1}{\alpha} - \dfrac{1}{\beta}\right)$ (*ii*) $\dfrac{a^2}{4} \tan\alpha\, e^{2\beta\cot\alpha}\,[e^{2\gamma\cot\alpha} - 1]$

2. (*i*) $\dfrac{\pi a^2}{8}$ (*ii*) $\dfrac{\pi a^2}{12}$ (*iii*) $\dfrac{\pi a^2}{16}$

(*iv*) $\dfrac{\pi a^2}{4n}$; total area $= \dfrac{\pi a^2}{4}$ or $\dfrac{\pi a^2}{2}$ according as n is an odd or even positive integer

5. $\dfrac{\pi a^2}{16}\left(\dfrac{\pi^2}{6} - 1\right)$

6.

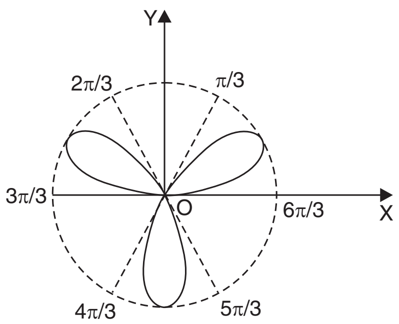


7. (*i*) $\dfrac{3}{2}\pi a^2$ (*ii*) $\dfrac{3}{2}\pi a^2$ **10.** $\dfrac{\pi}{2}(a^2 + b^2)$

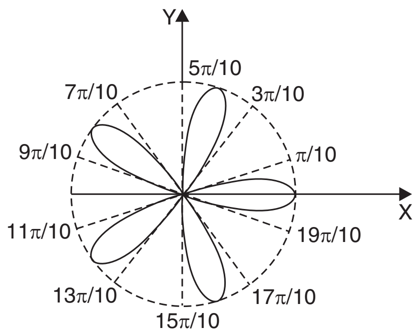
12. (*a*) $\dfrac{\pi a^2}{4}$ (*b*) $\dfrac{\pi a^2}{2}$ (*c*) $\dfrac{5a^2}{2}$.

Note. For the sake of convenience of the reader the shapes of the curves in question 2 (*ii*) and question 4 are being given below.

2. (*ii*)



4.



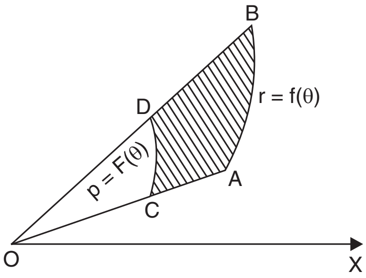
AREA BETWEEN TWO POLAR CURVES

To prove that the area bounded by the curves $r = f(\theta)$, $r = F(\theta)$ *and the radii vectors* $\theta = \alpha$, $\theta = \beta$ *is*

$$\int_\alpha^\beta \frac{1}{2}(r_1^2 - r_2^2)\,d\theta$$

where r_1 *is the 'r' of the outer curve and* r_2 *that of the inner curve.*

Let AB, CD be the curves $r = f(\theta)$, $r = F(\theta)$ and OCA, ODB the radii vectors $\theta = \alpha$, $\theta = \beta$ respectively.



Then area CABD

$$= \text{area OAB} - \text{area OCD}$$

$$= \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_\alpha^\beta \tfrac{1}{2}[F(\theta)]^2\,d\theta$$

$$= \int_\alpha^\beta \tfrac{1}{2}\,[[f(\theta)]^2 - [F(\theta)]^2]\,d\theta$$

$$= \int_\alpha^\beta \tfrac{1}{2}\,(r_1^2 - r_2^2)\,d\theta$$

where $r_1 = [f(\theta)]$ is the value of r of the outer curve and $r_2 = [F(\theta)]$ is that of the inner curve.

Example 1. *Find the area common to the circle $r = a$ and the cardioid $r = a\,(1 + \cos\theta)$.*

Sol. Equations of the two curves are

$r = a$ ...(1) (circle with pole as centre and radius a)

and $r = a\,(1 + \cos\theta)$ ...(2) (cardioid)

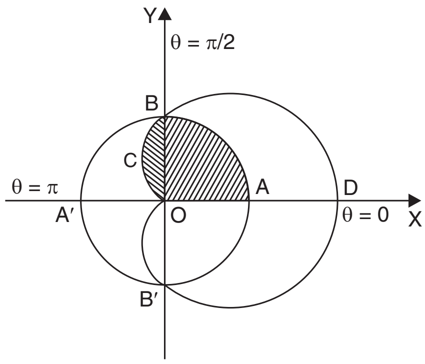
Solving (1) and (2) [To find points of Intersection], by putting $r = a$, from (1) in (2), we have $a = a(1 + \cos\theta)$

or $\cos\theta = 0$ or $\theta = \pm\dfrac{\pi}{2}$

i.e., two curves cut each other at points where $\theta = \pm\dfrac{\pi}{2}$.

∴ Required area = 2 (the shaded area) = 2 [Area OABO + Area OBCO] ...(3)

Now area OABO



$$= \int_0^{\pi/2} \frac{1}{2}\,r^2\,d\theta \text{ for } r = a$$

$$= \int_0^{\pi/2} \frac{1}{2}\,a^2\,d\theta$$

$$= \frac{a^2}{2}\Big[\theta\Big]_0^{\pi/2} = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4} \qquad \text{...(4)}$$

And area OBCO

$$= \int_{\pi/2}^{\pi} \frac{1}{2}\,r^2\,d\theta \text{ for } r = a(1 + \cos\theta)$$

$$= \frac{1}{2}a^2 \int_{\pi/2}^{\pi} (1 + \cos\theta)^2\,d\theta$$

$$= \frac{1}{2}a^2 \int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{a^2}{2}\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\int_{\pi/2}^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi} = \frac{a^2}{2}\left[\frac{3\pi}{2} - \left(\frac{3\pi}{4} + 2\right)\right] = \frac{a^2}{8}\,(3\pi - 8) \qquad \text{...(5)}$$

Putting the values from (4) and (5) in (3), the required area

$$= 2\left[\frac{\pi a^2}{4} + \frac{a^2}{8}(3\pi - 8)\right] = a^2\left(\frac{5\pi}{4} - 2\right).$$

EXERCISE 6.5

NOTES

1. Prove that the area common to the circles $r = a\sqrt{2}$ and $r = 2a \cos \theta$ is $a^2(\pi - 1)$.
2. Prove that the area of the region included between the cardioids $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$ is $\frac{a^2}{2}(3\pi - 8)$.
3. Find the area outside the circle $r = 2a \cos \theta$ and inside the cardioid $r = a(1 + \cos \theta)$.
4. Find the area inside the circle $r = \sin \theta$ and outside the cardioid $r = 1 - \cos \theta$.

Answers

3. $\frac{\pi a^2}{2}$ **4.** $1 - \frac{\pi}{4}$.

MISCELLANEOUS EXERCISE

1. (*a*) Find the area of the segment cut off from the parabola $y^2 = 4x$ by the straight line $2x - 3y + 4 = 0$.

   (*b*) Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$.
2. Show that the area included between one of the branches of the curve $x^2y^2 = a^2(x^2 + y^2)$ and its asymptotes is equal to that of the square whose side is a.

   $\left[\textbf{Hint.} \text{ Area in first quadrant} = \int (y_1 - y_2)\, dx = \int_a^{\infty} \left(\frac{ax}{\sqrt{x^2 - a^2}} - a\right) dx.\right]$
3. Show that the area of the loop of the curve $a^2y^2 = x^2(2a - x)(x - a)$ is $\frac{3}{8}a^2\pi$.
4. Show that the area of the loop of the curve $y^2 = 3x^3 - x^4 - 2x^2$ is $\frac{3\pi}{8}$.

   [**Hint.** Equation of the curve is $y^2 = 3x^3 - x^4 - 2x^2$

   $= x^2(-x^2 + 3x - 2)$

   $= x^2(-x^2 + x + 2x - 2) = x^2[-x(x - 1) + 2(x - 1)]$

   or $y^2 = x^2(x - 1)(2 - x)$

   Now it is Q. No. 3 with $a = 1$].
5. Show that the area enclosed by the curves $xy^2 = a^2(a - x)$ and $(a - x)y^2 = a^2x$ is $(\pi - 2)a^2$.

   $\left[\textbf{Hint.} \text{ The point of intersection is } x = \frac{a}{2}\right]$.
6. Find the area of the loop of the curve $x = \frac{a \sin 3\theta}{\sin \theta}$, $y = \frac{a \sin 3\theta}{\cos \theta}$.

   [**Hint.** Dividing $\tan \theta = \frac{y}{x}$ $\therefore$ $\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$

   Eliminating θ, equation of the curve is $y^2(a + x) = x^2(3a - x)$].
7. Find the area of a loop of the curve $y^2x + (x + a)^2(x + 2a) = 0$.

   $\left[\textbf{Hint.} \text{ Area of loop} = 2\int_{-2a}^{-a} y\, dx = 2\int_{-2a}^{-a} (x + a)\sqrt{\frac{x + 2a}{-x}}\, dx. \text{ Put } x = -2a \sin^2 \theta.\right]$
8. Prove that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of its loop.
9. Show that the area of a loop of the curve $r = \sqrt{3} \cos 3\theta + \sin 3\theta$ is $\frac{\pi}{3}$.

NOTES

10. Find the area of the ellipse $\frac{l}{r} = 1 + e\cos\theta$.

11. Find the ratio of the two parts into which the parabola $2a = r(1 + \cos\theta)$ divides the area of the cardioid $r = 2a(1 + \cos\theta)$.

12. Show that the ratio of the area of the larger to the area of the smaller loop of the curve $r = \frac{1}{2} + \cos 2\theta$ is $\frac{(4\pi + 3\sqrt{3})}{(2\pi - 3\sqrt{3})}$.

13. Show that the area of the region enclosed between the two loops of the curve $r = a(1 + 2\cos\theta)$ is $a^2(\pi + 3\sqrt{3})$.

14. Show that the area of the segment cut off from the parabola $y^2 = 2x$ by the straight line $y = 4x - 1$ is $\frac{9}{32}$.

15. Find the area bounded by the parabola $x^2 = 8y$ and the line $x - 2y + 8 = 0$.

16. Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is $\frac{1}{2}ab(\theta - e\sin\theta)$ where e is the eccentricity of the ellipse and θ is the eccentric angle of the point to which the radius vector is drawn.

17. If A is the area contained between the catenary $y = c\cosh\frac{x}{c}$, the two axes and an ordinate at the extremity of the arc s, show that $A = cs$; s being measured from the vertex.

18. Prove that the area of a loop of the curve $r = a\cos 3\theta + b\sin 3\theta$ is $\frac{1}{12}\pi(a^2 + b^2)$.

19. Find the area of the loop of the curve $x^4 + 3x^2y^2 + 2y^4 = a^2xy$.

Answers

1. (a) $\frac{1}{3}$ (b) $\frac{16}{5}$ **6.** $3\sqrt{3}\,a^2$

7. $2a^2\left(1 - \frac{\pi}{4}\right)$ **10.** $\frac{\pi l^2}{(1 - e^2)^{3/2}}$ **11.** $\frac{9\pi - 16}{9\pi + 16}$

15. $\frac{76}{3}$ **19.** $\frac{a^2}{4}\log 2$.

7. BETA AND GAMMA FUNCTIONS

STRUCTURE

- Beta Function
- Gamma Function

In this chapter, we are going to study Beta and Gamma functions, their properties and relationship

BETA FUNCTION

We have seen in example 7 of chapter 10 that the integral

$$\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$$

is convergent for all positive values of m and n. *This integral, obviously a function of m and n, is called a* **Beta function** *and is denoted by* $B(m, n)$, *i.e.*,

$$B(m, n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx, \quad \forall\ m > 0,\ n > 0$$

Beta function is also called the **First Eulerian Integral***.

For example,

(*i*) $\int_0^1 x^4\,(1-x)^5\,dx$ is a Beta function B(5, 6).

The integral can be easily evaluated by expanding $(1-x)^5$.

(*ii*) $\int_0^1 x^{2/3}(1-x)^6\,dx$ is a Beta function $B\left(\frac{5}{3}, 7\right)$

The integral can be easily evaluated by expanding $(1-x)^6$.

(*iii*) $\int_0^1 x^3\,(1-x)^{4/3}\,dx = B\left(4, \frac{7}{3}\right)$

This integral can be evaluated using the symmetric property (given below) of the beta function.

*After the name of a great mathematician and physicist L. Euler (1707 – 1783), who was a member of St. Petersburg Academy of Sciences.

NOTES

Note that B (m, n) can be easily evaluated if at least one of m and n is a positive integer.

Symmetry of Beta Function. $B(m, n) = B(n, m)$.

Proof. By definition,

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx,\ m > 0,\ n > 0$$

Changing x to $1 - x$, we have

$$B(m, n) = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\, dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1}\, dx = B(n, m).$$

Note. Integral of example (*iii*) above, can be written as

$$B\left(4, \frac{7}{3}\right) = B\left(\frac{7}{3}, 4\right) = \int_0^1 x^{4/3} (1-x)^3\, dx$$

and can be easily integrated.

Example 1. *Express* $\int_0^2 (8 - x^3)^{-1/3}\, dx$ *in terms of a Beta function.*

Sol. Let $\quad I = \int_0^2 (8 - x^3)^{-1/3}\, dx \qquad ...(1)$

Put $\quad x^3 = 8z$

$\therefore \quad x = 2z^{1/3}$ and $dx = \frac{2}{3} z^{-2/3}\, dz$

Also, when $x = 0$, $z = 0$; when $x = 2$, $z = 1$

$\therefore$ (1) becomes

$$I = \int_0^1 (8 - 8z)^{-1/3} \cdot \frac{2}{3} z^{-2/3}\, dz = 8^{-1/3} \cdot \frac{2}{3} \int_0^1 (1-z)^{-1/3} z^{-2/3}\, dz$$

$$= \frac{1}{3} \int_0^1 (1-x)^{-1/3} x^{-2/3}\, dx \qquad \left[\because \int_a^b f(z)\, dz = \int_a^b f(x)\, dx\right]$$

$$= \frac{1}{3} \int_0^1 x^{-2/3} (1-x)^{-1/3}\, dx$$

$$= \frac{1}{3} B\left(\frac{-2}{3} + 1, \frac{-1}{3} + 1\right) = \frac{1}{3} B\left(\frac{1}{3}, \frac{2}{3}\right).$$

Example 2. *Evaluate* $\int_a^b (x - a)^{m-1} (b - x)^{n-1}\, dx,\ m > 0,\ n > 0.$

Sol. Put $x = a + (b - a)z$ so that $dx = (b - a)\, dz$

$$\therefore \quad \int_a^b (x-a)^{m-1} (b-x)^{n-1}\, dx$$

$$= \int_0^1 [(b-a)z]^{m-1} [b - a - (b-a)z]^{n-1} (b-a)\, dz$$

$$= (b-a)^{m+n-1} \int_0^1 z^{m-1} (1-z)^{n-1}\, dz$$

$$= (b - a)^{m+n-1} \int_0^1 x^{m-1} (1 - x)^{n-1} \, dx$$

$$= (b - a)^{m+n-1} \,.\, B(m, a).$$

Example 3. *Show that*

$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} \, dx = \frac{1}{(a+b)^m a^n} \, B(m, n).$$

Sol. Put $\dfrac{x}{a+bx} = \dfrac{z}{a+b\,.\,1}$

Differentiating w.r.t. x, we have

$$\frac{(a+bx)\,.\,1 - x\,.\,b}{(a+bx)^2} dx = \frac{dz}{a+b} \qquad \Rightarrow \qquad \frac{dx}{(a+bx)^2} = \frac{dz}{a(a+b)}$$

Also, when $x = 0$, $z = 0$; when $x = 1$, $z = 1$.

$$\therefore \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} \, dx = \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} \left(\frac{1-x}{a+bx}\right)^{n-1} . \frac{1}{(a+bx)^2} dx$$

$$= \int_0^1 \left(\frac{z}{a+b}\right)^{m-1} \left(\frac{1-z}{a}\right)^{n-1} . \frac{dz}{a(a+b)}$$

$$\left[\because \frac{x}{a+bx} = \frac{z}{a+b} \qquad \Rightarrow \qquad (a + b)x = az + bxz\right.$$

$$\Rightarrow \qquad (a + b - bz)x = az \qquad \Rightarrow \qquad x = \frac{az}{a+b-bz}$$

$$\Rightarrow \qquad \frac{1-x}{a+bx} = \frac{1 - \dfrac{az}{a+b-bz}}{a + \dfrac{abz}{a+b-bz}} = \frac{a+b-bz-az}{a^2 + ab - abz + abz}$$

$$\left. = \frac{(a+b)(1-z)}{a(a+b)} = \frac{1-z}{a}\right]$$

$$= \frac{1}{a^n (a+b)^m} \int_0^1 z^{m-1} (1 - z)^{n-1} \, dz = \frac{B(m, n)}{(a+b)^m \,.\, a^n}.$$

Example 4. *Show that*

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = B(m, n), \; m > 0, \; n > 0.$$

Sol. Put $\dfrac{x}{1+x} = z$

$$\Rightarrow \qquad x = \frac{z}{1-z}$$

$$\therefore \qquad dx = \frac{(1-z)\,.\,1 - z\,(-1)}{(1-z)^2} dz = \frac{dz}{(1-z)^2}$$

Also, $\quad 1 + x = 1 + \dfrac{z}{1-z} = \dfrac{1}{1-z}$

Further, when $x = 0$, $z = 0$;

when $x \to \infty$, $z = \lim\limits_{x \to \infty} \dfrac{x}{1+x} = \lim\limits_{x \to \infty} \dfrac{1}{1 + \dfrac{1}{x}} = 1$

NOTES

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{\left(\frac{z}{1-z}\right)^{m-1}}{\left(\frac{1}{1-z}\right)^{m+n}} \cdot \frac{dz}{(1-z)^2}$$

$$= \int_0^1 z^{m-1}\,(1-z)^{n-1}\,dz = \text{B}(m, n).$$

Remark. Result of the above example is also used in solving questions.

Example 5. *Show that*

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = B(m, n),\ m > 0,\ n > 0.$$

Sol.
$$\text{B}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} \qquad \ldots(1)$$

Consider $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Put $\quad x = \frac{1}{t}$

$\therefore \quad dx = -\frac{1}{t^2}\,dt$

Also, when $x = 1$, $t = 1$; when $x \to \infty$, $t \to 0$

$$\therefore \quad \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int_1^0 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \qquad \ldots(2)$$

Using (2), (1) becomes

$$\text{B}(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Example 6. *If p, q are positive, show that*

(i) $\frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p}$ $\qquad$ *(ii)* $B(p, q) = B(p+1, q) + B(p, q+1)$.

Sol. *(i)* $\quad \text{B}(p, q+1)^* = \int_0^1 x^{p-1}\,(1-x)^q\,dx$

*To change a Beta function to integral decrease each part by 1, *i.e.*,

$$\text{B}(p, q) = \int_0^1 x^{p-1}\,(1-x)^{q-1}\,dx.$$

Integrating by parts taking $(1-x)^q$ as first function, we have

[∵ on comparing the two integrals B(p, q + 1) and B(p + 1, q), we find that power of $1-x$ is to be decreased by 1]

$$B(p, q+1) = \left[(1-x)^q \frac{x^p}{p}\right]_0^1 - \int_0^1 -q(1-x)^{q-1}\frac{x^p}{p}\,dx$$

$$= 0 + \frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx = \frac{q}{p}B(p+1, q)$$

$$\Rightarrow \qquad \frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p}$$

(*ii*)

$$\text{R.H.S.} = B(p+1, q) + B(p, q+1)$$

$$= \int_0^1 x^p(1-x)^{q-1}\,dx + \int_0^1 x^{p-1}(1-x)^q\,dx$$

$$= \int_0^1 [x^p(1-x)^{q-1} + x^{p-1}(1-x)^q\,dx$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}(x+1-x)\,dx$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = B(p, q) = \text{L.H.S.}$$

If m, n are positive integers, prove that

$$B(m, n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}.$$

Proof. $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$

Integrating by parts (taking x^{m-1} as first function), we have

$$B(m, n) = \left[x^{m-1}\frac{(1-x)^n}{-n}\right]_0^1 - \int_0^1 (m-1)\,x^{m-2}\frac{(1-x)^n}{-n}\,dx$$

$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\,dx$$

$$\Rightarrow \qquad B(m, n) = \frac{m-1}{n}B(m-1, n+1)$$

Changing m to $m-1$ and n to $n+1$, we have

$$B(m-1\,;\,n+1) = \frac{m-2}{n+1}B(m-2, n+2)$$

Similarly, $B(m-2, n+2) = \dfrac{m-3}{n+2}B(m-3, n+3)$

Repeating the above process $m-2$ times, we have

$$B(2, n+m-2) = \frac{1}{n+m-2}B(1, n+m-1)$$

Multiplying the above $m-1$ equations, we have

$$B(m, n) = \frac{(m-1)(m-2)\ldots\ldots 1}{n(n+1)\ldots\ldots(n+m-2)}B(1, n+m-1) \qquad \ldots(1)$$

NOTES

But, $$\mathrm{B}(1, n+m-1) = \int_0^1 x^{1-1} (1-x)^{n+m-1-1}\, dx$$

$$= \int_0^1 (1-x)^{n+m-2}\, dx$$

$$= \left[\frac{(1-x)^{n+m-1}}{-(n+m-1)}\right]_0^1 = \frac{1}{n+m-1}$$

Substituting this value in (1), we have

$$\mathrm{B}(m, n) = \frac{(m-1)(m-2)\ldots\ldots 1}{n(n+1)\ldots\ldots(n+m-2)(n+m-1)} \qquad \ldots(2)$$

$$= \frac{(m-1)\,!}{(n+m-1)(n+m-2)\ldots\ldots(n+1)n}$$

$$= \frac{(m-1)\,!\,(n-1)\,!}{(n+m-1)(n+m-2)\ldots\ldots(n+1)n\,.\,(n-1)\,!}$$

$$= \frac{(m-1)\,!\,(n-1)\,!}{(n+m-1)\,!}.$$

Cor. Putting $m = 1$, in the above result, we have

$$\mathrm{B}(1, n) = \frac{1}{n}.$$

EXERCISE 7.1

1. Express in terms of Beta functions :

(*i*) $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx$ (*ii*) $\int_0^2 x^3(8-x^3)^{-1/3}\, dx$

(*iii*) $\int_0^1 \frac{x^2}{1-x^5}\, dx$ (*iv*) $\int_0^1 x^{l-1}(1-x^2)^{m-1}\, dx$

(*v*) $\int_0^p x^m (p^a - x^a)^n\, dx.$ *(K.U. BCA (II) 2001)*

[Hint. (*i*) Put $x^5 = z$ (*ii*) Put $x^3 = 8z$ (*v*) Put $x^a = p^a z$.]

2. Prove that $\int_0^a (a-x)^{m-1} x^{n-1}\, dx = a^{m+n-1}\, \mathrm{B}(m, n)$.

3. Using the property $\mathrm{B}(m, n) = \mathrm{B}(n, m)$, evaluate

$$\int_0^1 x^3(1-x)^{4/3}\, dx.$$

4. Show that $\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$, m, n being positive integers.

[Hint. $\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $\sqrt{x} = \sin\theta$, etc.]

5. Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \frac{\mathrm{B}(m, n)}{a^n(1+a)^m}$.

[Hint. See example 3.]

6. Show that

(*i*) $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = 2B(m, n),\ m > 0,\ n > 0$

(*ii*) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx = 0,\ m > 0,\ n > 0.$

GAMMA FUNCTION

We have seen in Example 11 of chapter 10, that the integral $\int_0^\infty x^{n-1} e^{-x}\,dx$ is convergent for each positive n. *The integral* $\int_0^\infty x^{n-1} e^{-x}\,dx$ *for* $n > 0$, *is obviously a function of n and is called a Gamma function of n. It is denoted by* $\Gamma(n)$, *i.e.*,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx, \quad \forall \quad n > 0.$$

Gamma function is also called the Second Eulerian Integral.

For example,

(*i*) $\int_0^\infty x^5 e^{-x}\,dx$ is a Gamma function $\Gamma(6)$

(*ii*) $\int_0^\infty x^{5/3} e^{-x}\,dx$ is a Gamma function $\Gamma\left(\frac{8}{3}\right)$.

Recurrence Formula for Gamma Function Γ(n). *To prove that* $\Gamma(n) = (n-1)\Gamma(n-1)$, *when* $n > 1$.

Proof. By definition,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx.$$

Integrating by parts,

$$= \left[x^{n-1} \cdot \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty (n-1)x^{n-2}\frac{e^{-x}}{-1}\,dx$$

$$= -\left[\lim_{x\to\infty} \frac{x^{n-1}}{e^x} - 0\right] + (n-1)\int_0^\infty x^{n-2} e^{-x}\,dx$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\,dx \qquad \left[\because \lim_{x\to\infty}\frac{x^{n-1}}{e^x} = 0 \text{ for } n > 0\right]$$

$$= (n-1)\Gamma(n-1)$$

Thus, $\Gamma(n) = (n-1)\Gamma(n-1)$.

Note. $\Gamma(n) > 0$ always.

Cor. *If n is a positive integer, then* $\Gamma(n) = (n-1)!$.

Proof. $\Gamma(n) = (n-1)\,\Gamma(n-1)$

Changing n to $n-1$, $n-2$, ..., successively $n-1$ times.

NOTES

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$
$$\Gamma(n-2) = (n-3)\,\Gamma(n-3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Gamma(2) = 1.\,\Gamma(1)$$

Multiplying, we have

$$\Gamma(n) = (n-1)(n-2)\ldots\ldots 1\,.\,\Gamma(1) \qquad \ldots(1)$$

But, $\Gamma(1) = \int_0^\infty x^{1-1}\, e^{-x}\, dx = \int_0^\infty e^{-x}\, dx$

$$= \lim_{t \to \infty} \int_0^t e^{-x}\, dx = \lim_{t \to \infty} -[e^{-t} - 1] = 1$$

Hence from (1), $\Gamma(n) = (n-1)\,!$

For example, $\Gamma(4) = 3\,! = 6$.

Relation between Beta and Gamma Functions. *To prove that*

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \text{ where } m > 0,\ n > 0.$$

Proof. To prove the given relation, we first prove

$$\Gamma(n) = z^n \int_0^\infty x^{n-1}\, e^{-zx}\, dx$$

Putting $x = az$ in $\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$, we have

$$\Gamma(n) = \int_0^\infty (az)^{n-1}\, e^{-az}\, a\, dz = a^n \int_0^\infty z^{n-1}\, e^{-az}\, dz$$

Replacing z by x,

$$= a^n \int_0^\infty x^{n-1}\, e^{-ax}\, dx$$

Replacing a by z, we have

$$\Gamma(n) = z^n \int_0^\infty x^{n-1}\, e^{-zx}\, dx \qquad \ldots(1)$$

$$\Rightarrow \qquad \Gamma(n) = \int_0^\infty x^{n-1}\, z^n\, e^{-zx}\, dx$$

Multiplying both sides by $e^{-z}\, z^{m-1}$, we have

$$\Gamma(n)\,.\,e^{-z}\, z^{m-1} = \int_0^\infty x^{n-1}\, z^{n+m-1}\, e^{-z(1+x)}\, dx$$

Integrating both sides w.r.t. z between the limits 0 to ∞, we have

$$\int_0^\infty G(n)\,.\,e^{-z}\, z^{m-1}\, dz = \int_0^\infty \left[\int_0^\infty x^{n-1}\, z^{n+m-1}\, e^{-z(1+x)}\, dx\right] dz$$

$$\Gamma(n) \int_0^\infty e^{-z}\, z^{m-1}\, dz = \int_0^\infty x^{n-1} \left[\int_0^\infty z^{n+m-1}\, e^{-z(1+x)}\, dz\right] dx.$$

[Put $z(1+x) = y$, etc. on R.H.S.]

$$\Rightarrow \quad \Gamma(n)\Gamma(m) = \int_0^\infty x^{n-1} \left[\int_0^\infty \frac{y^{m+n-1}\, e^{-y}\, dy}{(1+x)^{m+n}}\right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left[\int_0^\infty y^{m+n-1}\, e^{-y}\, dy\right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, [\Gamma(m+n)] dx$$

$$= \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \Gamma(m+n)\, B(m, n)$$

$$\left(\because \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} = B(m, n) \text{ Example 4 after 11.1.1}\right)$$

Hence $\quad B(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$

Cor. 1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof. Substituting $m = \frac{1}{2}$ and $n = \frac{1}{2}$ in the relation

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{ we have}$$

$$B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}$$

$$= \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 \qquad [\because \Gamma(1) = 1]$$

$$\Rightarrow \quad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \int_0^1 x^{\frac{1}{2}-1} (1-x)^{\frac{1}{2}-1}\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\,\sqrt{1-x}}\, dx \qquad \ldots(1)$$

Let $\quad x = \sin^2 \theta$

$\therefore \quad dx = 2 \sin \theta \cos \theta\, d\theta$

Also, when $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \dfrac{\pi}{2}$

$\therefore$ from (1),

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \int_0^{\pi/2} \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}}\, .\, 2 \sin q \cos q\, dq$$

$$= \int_0^{\pi/2} 2\, .\, d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

$$\Rightarrow \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

Cor. 2. $\displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$

Proof. Let $\quad I = \displaystyle\int_0^\infty e^{-x^2}\, dx$

Put $\quad x^2 = z$

$$\Rightarrow \quad 2x\, dx = dz \quad \Rightarrow \quad dx = \frac{dz}{2x} = \frac{1}{2\sqrt{z}}\, dz$$

Also, when $x = 0$, $z = 0$; when $x \to \infty$, $z \to \infty$

$$\therefore \qquad I = \int_0^\infty e^{-z} \frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^\infty \frac{e^{-x}}{\sqrt{z}}\, dz$$

NOTES

$$= \frac{1}{2}\int_0^\infty z^{-\frac{1}{2}} e^{-z}\, dz$$

$$= \tfrac{1}{2}\Gamma(-\tfrac{1}{2}+1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi} \qquad \text{(By Cor. 1)}$$

Cor. 3. $\displaystyle\int_{-\infty}^0 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

One can prove the result by substituting $x = -z$.

Cor. 4. $\displaystyle\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$.

For, $\displaystyle\int_{-\infty}^\infty e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx = 2 \,.\, \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

$$\left[\because \text{ if } (f)\, x \text{ is an even function of } x, \text{ then } \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx\right]$$

Example 7. *Show that* $\displaystyle\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{1}{3}\sqrt{\pi}$.

Sol. Put $x^3 = z$* so that $dx = \dfrac{dz}{3x^2}$

Also, when $x = 0$, $z = 0$; when $x \to \infty$, $z \to \infty$.

$$\therefore \qquad \int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \int_0^\infty \sqrt{x}\, e^{-z} \frac{dz}{3x^2}$$

$$= \frac{1}{3}\int_0^\infty x^{-3/2} e^{-z}\, dz = \frac{1}{3}\int_0^\infty (z^{1/3})^{-3/2} e^{-z}\, dz$$

$$= \frac{1}{3}\int_0^\infty z^{-1/2} e^{-z} = \frac{1}{3}\Gamma(\tfrac{1}{2}) = \frac{1}{3}\sqrt{\pi} \qquad (\because \ \Gamma(\tfrac{1}{2}) = \sqrt{\pi})$$

Example 8. *Show that* $\dfrac{B(m+2, n-2)}{B(m, n)} = \dfrac{m(m+1)}{(n-1)(n-2)}$.

Sol.

$$\frac{B(m+2, n-2)}{B(m, n)} = \frac{\dfrac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m+2+n-2)}}{\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$= \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m)\Gamma(n)} = \frac{(m+1)m\Gamma(m)\Gamma(n-2)}{\Gamma(m)(n-1)(n-2)\Gamma(n-2)}$$

$$[\because \ \Gamma(n) = (n-1)\,\Gamma(n-1)]$$

$$= \frac{m(m+1)}{(n-1)(n-2)}.$$

*In an integral of the type $\int x^n e^{-f(x)}\, dx$, put $f(x) = z$.

Example 9. *Evaluate*

(*i*) $\int_0^\infty e^{-4x} x^{3/2}\, dx$ (*ii*) $\int_0^1 x^4 (1-x)^3\, dx$ (*iii*) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$.

Sol. (*i*) Putting $4x = y$, we have $dx = \frac{1}{4}\, dy$.

Also, when $x = 0$, $y = 0$; when $x \to \infty$, $y \to \infty$

$$\therefore \quad \int_0^\infty e^{-4x} x^{3/2}\, dx = \frac{1}{32} \int_0^\infty e^{-y} y^{5/2-1}\, dy$$

$$= \frac{1}{32}\Gamma\left(\frac{5}{2}\right) = \frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{128}\sqrt{\pi}.$$

(*ii*)
$$\int_0^1 x^4 (1-x)^3\, dx = B(5, 4)$$

$$= \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{4\,!\,3\,!}{8\,!} = \frac{1}{280}.$$

(*iii*) Putting $x = 2y$ so that $dx = 2dy$

Also, when $x = 0$, $y = 0$; when $x = 2$, $y = 1$

$$\therefore \quad \int_0^2 \frac{x^2\,.\,dx}{\sqrt{2-x}} = 4\sqrt{2}\int_0^1 y^2(1-y)^{-1/2}\, dy$$

$$= 4\sqrt{2}\; B(3, \tfrac{1}{2}) = 4\sqrt{2}\;\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{7}{2}\right)} = \frac{4\sqrt{2}\,.\,2\,!\,\Gamma(\frac{1}{2})}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)} = \frac{64\sqrt{2}}{15}.$$

Example 10. *Evaluate*

(*i*) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$ (*ii*) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$.

Sol. (*i*)
$$I = \int_0^\infty \frac{x^6(1-x^6)}{(1+x)^{24}}\, dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\, dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\, dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\, dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\, dx$$

$$= B(9, 15) - B(15, 9) = 0 \qquad [\because \; B(m, n) = B(n, m)]$$

(*ii*)
$$I = \int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$$

$$= \int_0^\infty \frac{x^4}{(1+x)^{15}}\, dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\, dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\, dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\, dx$$

$$= B(5, 10) + B(10, 5) = 2B(5, 10)$$

$$= \frac{2\Gamma(5)\,.\,\Gamma(10)}{\Gamma(5+10)} = \frac{2\,.\,4\,!\,.\,9\,!}{14\,!} = \frac{1}{5005}.$$

NOTES

To evaluate $\int_0^{\pi/2} \sin^p x \cos^q x\, dx,\ p > -1,\ q > -1.$

Proof. Put $\sin^2 x = y$ so that $2 \sin x \cos x\, dx = dy$.

Also, $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-y}$

$$\therefore \qquad dx = \frac{dy}{2 \sin x \cos x} = \frac{dy}{2\sqrt{y}\,\sqrt{1-y}}$$

Also, when $x = 0,\ y = 0$; when $x = \frac{\pi}{2}$, $y = 1$

Therefore,

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \int_0^1 (\sqrt{y})^p\, (\sqrt{1-y})^q\, \frac{2}{2\sqrt{y}\,\sqrt{1-y}}\, dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{p-1}{2}}\, (1-y)^{\frac{q-1}{2}}\, dy$$

$$= \frac{1}{2}\, \mathrm{B}\left(\frac{p-1}{2}+1, \frac{q-1}{2}+1\right)$$

$$= \frac{1}{2}\, \mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2}\, \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

Thus,

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)},\ p > -1,\ q > -1.$$

Example 11. *Evaluate*

(*i*) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} dx$ (*ii*) $\int_0^{\pi/2} \sin^5 x\, dx.$

Sol. (*i*) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} dx$ $\left(\text{Here } p = 3, q = \frac{5}{2}\right)$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{\frac{5}{2}+1}{2}\right)}{2\Gamma\left(\frac{3+1}{2}+\frac{\frac{5}{2}+1}{2}\right)} = \frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{2\Gamma\left(\frac{15}{4}\right)} = \frac{1\,.\,\Gamma\left(\frac{7}{4}\right)}{2\,.\,\frac{11}{4}\,\frac{7}{4}\,\Gamma\left(\frac{7}{4}\right)} = \frac{8}{77}.$$

(*ii*) $\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \sin^5 x \cos^0 x\, dx$ (Here $p = 5,\ q = 0$)

$$= \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{5+1}{2}+\frac{0+1}{2}\right)} = \frac{\Gamma(3)\,\Gamma(\frac{1}{2})}{2\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{2\,!\,\Gamma(\frac{1}{2})}{2\,.\,\frac{5}{2}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)} = \frac{8}{15}.$$

Duplication Formula. *To prove that*

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m).$$

Proof. We know

$$B(m, m) = \int_0^1 x^{m-1}\,(1-x)^{m-1}\,dx \qquad \ldots(1)$$

Putting $x = \sin^2\theta$, we have

$$dx = 2\sin\theta\cos\theta\,d\theta$$

Also, when $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \frac{\pi}{2}$

∴ (1) becomes

$$B(m, m) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}\,(1-\sin^2\theta)^{m-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta = 2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta \qquad \ldots(2)$$

Putting $2\theta = \phi$, we have $d\theta = \frac{1}{2}\,d\phi$.

Also, when $\theta = 0$, $\phi = 0$; when $\theta = \frac{\pi}{2}$, $\phi = \pi$

∴ from (2),

$$B(m, m) = \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi \quad (\because\ \sin^{2m-1}(\pi-\phi) = \sin^{2m-1}\phi)$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}\phi\,.\,\cos^0\phi\,d\phi$$

$$= \frac{2}{2^{2m-1}}\,\frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2m-1+1}{2}+\frac{0+1}{2}\right)} = \frac{1}{2^{2m-1}}\,\frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$\Rightarrow \qquad \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)} = \frac{1}{2^{2m-1}}\,\frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

Cancelling $\Gamma(m)$ and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have

$$\frac{\Gamma(m)}{\Gamma(2m)} = \frac{\sqrt{\pi}}{2^{2m-1}\,\Gamma(m+\frac{1}{2})}$$

$$\Rightarrow \qquad \Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$$

NOTES

Example 12. *Prove that*

$$B(m, m) \,.\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi\, m^{-1}}{2^{4m-1}}.$$

Sol. L.H.S. $= B(m, m).\ B(m + \frac{1}{2},\ m + \frac{1}{2})$

$$= \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} \cdot \frac{\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{1}{2}+m+\frac{1}{2})}$$

$$= \frac{[\Gamma(m)\Gamma(m+\frac{1}{2})]^2}{\Gamma(2m)\,.\,\Gamma(2m+1)}$$

$$= \frac{[\Gamma(m)\Gamma(m+\frac{1}{2})]^2}{\Gamma(2m)\,.\,2m\,\Gamma(2m)} \qquad [\because\ \Gamma(n) = (n-1)\Gamma(n-1)]$$

$$= \left(\frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}\right)^2 \cdot \frac{1}{2m\,[\Gamma(2m)]^2} \qquad \text{[by Duplication Formula]}$$

$$= \frac{\pi}{2^{4m-2}\,.\,2m} = \frac{\pi\, m^{-1}}{2^{4m-1}}.$$

EXERCISE 7.2

1. Show that

(*i*) $\int_0^\infty x^3\, e^{-x}\, dx = 6$

(*ii*) $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{1}{3}\sqrt{\pi}$

(*iii*) $\int_0^\infty 4x^4\, e^{-x^4}\, dx = \Gamma\left(\frac{5}{4}\right)$

(*iv*) $\int_0^\infty x^6\, e^{-2x}\, dx = \frac{45}{8}$

(*v*) $\int_0^1 x^2\,(1-x)^3\, dx = \frac{1}{60}$

(*vi*) $\int_0^3 \frac{dx}{\sqrt{3x - x^2}} = \pi$

(*vii*) $\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)}.$

2. Express $\int_0^1 x^m\,(1-x^n)^p\, dx$ in terms of the Beta function and hence evaluate

$\int_0^1 x^5\,(1-x^3)^{10}\, dx.$

3. Prove that $\Gamma(n+1) = n\,\Gamma(n)$, where $n > 0$.

4. Prove that $\int_0^\infty e^{-ax}\, x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$, where $a > 0$, $n > 0$.

[**Hint.** Put $ax = z$.]

5. (*i*) Prove that $\int_0^\infty x^n e^{-a^2x^2}\, dx = \dfrac{1}{2a^{n+1}}\Gamma\left(\dfrac{n+1}{2}\right)$

Hence show that $\int_{-\infty}^\infty e^{-a^2x^2}\, dx = \dfrac{\sqrt{\pi}}{a}$.

[**Hint.** Put $a^2x^2 = z$.]

(*ii*) Prove that $\int_0^\infty e^{-a^2x^2}\, dx = \dfrac{\sqrt{\pi}}{2a}$, $a > 0$.

6. Show that $\int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \dfrac{\Gamma(m)\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}$.

[**Hint.** See Example 3.]

7. Show that $\int_0^1 \dfrac{dx}{\sqrt{1-x^n}} = \dfrac{\sqrt{\pi}\,\Gamma\left(\dfrac{1}{n}\right)}{n\,\Gamma\left(\dfrac{1}{n}+\dfrac{1}{2}\right)}$.

$\left[\textbf{Hint.}\text{ Put } x^n = z \text{ and use } \mathrm{B}(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.\right]$

8. (*i*) Prove that $\dfrac{\mathrm{B}(p, q+1)}{q} = \dfrac{\mathrm{B}(p+1, q)}{p} = \dfrac{\mathrm{B}(p, q)}{p+q}$, $p > 0$, $q > 0$.

(*ii*) Prove that $\dfrac{\mathrm{B}(m+1, n)}{\mathrm{B}(m, n)} = \dfrac{m}{m+n}$, $m > 0$, $n > 0$.

9. (*i*) Show that

$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n+2}{2}\right)}, \quad n > -1.$$

(*ii*) Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \dfrac{d\theta}{\sqrt{\sin\theta}} = \pi$.

10. Show that $\int_0^\infty \dfrac{x^a}{a^x}\, dx = \dfrac{\Gamma(a+1)}{(\log a)^{n+1}}$, if $a > 1$.

[**Hint.** $a^x = e^{x \log a}$. Put $x \log a = z$.]

11. Prove that $\mathrm{B}(m, m) = 2^{1-2m}\,\mathrm{B}(m, \tfrac{1}{2})$. [See solution of Duplication formula]

UNIT IV

NOTES

8. VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION

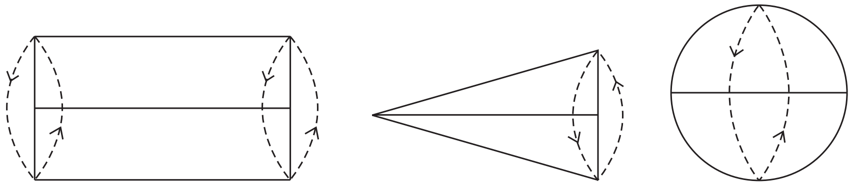
STRUCTURE

INTRODUCTION

A plane area when made to revolve about a fixed straight line lying in its own plane, generates **a solid of revolution** and its boundary generates a **surface of revolution.** The straight line about which the plane area is rotated is called the **axis of revolution.**

Every section of such a solid by a plane perpendicular to the axis of revolution is a circle having its centre on the axis.

For example (*a*) A right circular cylinder is generated by the revolution of a rectangle about one of its sides.



(a) (b) (c)

(*b*) A right circular cone is generated by the revolution of a right-angled triangle about its base.

(*c*) A sphere is generated when a semi-circle is rotated about its bounding diameter.

VOLUME FORMULAE FOR THE CARTESIAN EQUATIONS

If $y = f(x)$ be continuous, finite and a single valued function of x in the interval $[a, b]$ where a and b are finite and $a < b$, then the volume of solid generated by revolution about x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the bounded ordinates $x = a$, $x = b$ is $\int_a^b \pi y^2\, dx$.

Let AB be the curve $y = f(x)$ and CA, DB be the two ordinates $x = a$, $x = b$ respectively.

Let P (x, y) be *any* point on the curve AB.

Draw PM ⊥ OX. ∴ OM = x, PM = y.

Let V denote the volume of the solid generated by the revolution about x-axis of the area ACMP.

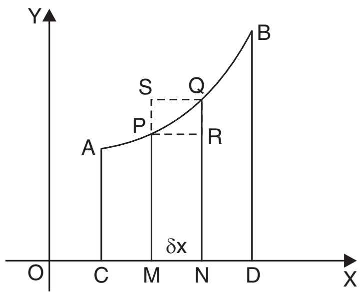
As x increases *i.e.* PM moves towards the right, V also increases.

∴ V is a function of x.

Let Q$(x + \delta x, y + \delta y)$ be another point on the curve in the immediate neighbourhood of P and NQ its ordinate.

Then the volume of the solid generated by the revolution about the x-axis of the area PMNQ is δV.

Complete the rectangle PRQS.



Then the volume of the solid generated by the revolution of the area PMNQ lies between the volumes of the circular cylinders generated by the rectangles PMNR and SMNQ *i.e.* δV lies between $\pi y^2\, \delta x$ and $\pi(y + \delta y)^2\, \delta x$.

or $$\pi y^2\, \delta x < \delta V < \pi(y + \delta y)^2\, \delta x$$

or $$\pi y^2 < \delta V/\delta x < \pi(y + \delta y)^2$$

Since y is a continuous function. ∴ $\delta y \to 0$ as $\delta x \to 0$.

∴ Taking limits as $\delta x \to 0$

$\underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta V}{\delta x}$ lies between πy^2 and a quantity which $\to \pi y^2$

$$\therefore \quad \frac{dV}{dx} = \pi y^2$$

$$\therefore \quad \int_a^b \pi y^2\, dx = \int_a^b \frac{dV}{dx}\, dx = [V]_a^b$$

= (volume V when $x = b$) – (volume V when $x = a$)

= volume generated by the revolution of the area ACDB – 0

Hence the volume of the solid generated by the revolution of the area ACDB about the x-axis is $\int_a^b \pi y^2\, dx$.

Note. In the above investigation, we observe that the ordinate increases continuously. The above result is still true when the ordinate decreases continuously or when the ordinate increases in some parts and decreases in the other parts. It is also assumed that curve does not cross the x axis, *i.e.* the axis of revolution.

NOTES

Cor. 1. Revolution about y-axis. *The volume of the solid generated by revolution about the* **y-axis** *of the area bounded by the curve x = f(y), the y-axis and the abscissae y = a, y = b is*

$$\int_a^b \pi x^2 \, dy.$$

The result follows immediately on interchanging x and y in the above proposition.

Cor. 2. Volume formulae for parametric equations. *If* $x = f(t)$ *and* $y = \phi(t)$ *are the parametric equations of a curve, then the volume of the solid generated by revolving the area about x-axis is*

$$\int \pi y^2 \frac{dx}{dt} . dt,$$

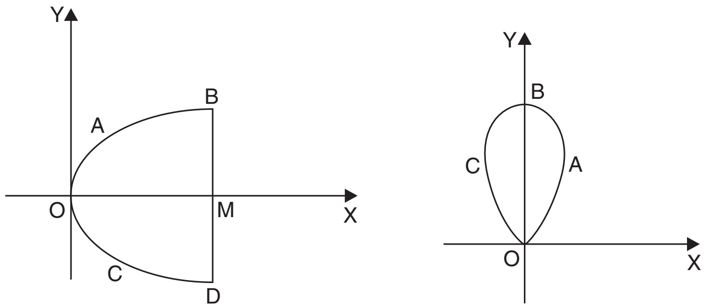
the limits of integration being so taken as to cover the whole area revolved and volume of the solid formed by revolving the area about y-axis is

$$\int \pi x^2 \frac{dy}{dt} . dt,$$

limits of integration being so taken as to cover the whole area rotated.

Note. An important observation. (*a*) If the generating curve *is symmetrical about the* x-axis, then the volume generated by the revolution of the area about the x-axis is the same as the volume generated by the revolution of its upper (or lower) half area.

Thus, the volume generated by the area DCOAB = vol. generated by the area OABM or OCDM.



(*b*) If the curve is *symmetrical about the y-axis* and the same curve be made to *revolve about the x-axis* (the curve lying on one side of x-axis), then the volume generated

= 2 × volume generated by the portion OAB lying in the first quadrant.

VOLUME FORMULAE FOR POLAR EQUATIONS

The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$:

(*i*) **about the initial line OX** ($\theta = 0$) $= \int_\alpha^\beta \frac{2}{3} \pi r^3 \sin\theta \, d\theta.$

(*ii*) **about the line OY** $\left(\theta = \frac{\pi}{2}\right) = \int_\alpha^\beta \frac{2}{3} \pi r^3 \cos\theta \, d\theta.$

The proofs of these formulae are beyond the scope of this book.

Note. The volume formulae of cartesian curves can also be used for polar curves by converting the polar equation to cartesian equation by using the relations $x = r \cos\theta$ and $y = r \sin\theta$.

$$\left(i.e.\ r^2 = x^2 + y^2,\ \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}\right).$$

Example 1. *Find the volume of the prolate spheroid generated by the revolution of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *about the x-axis.*

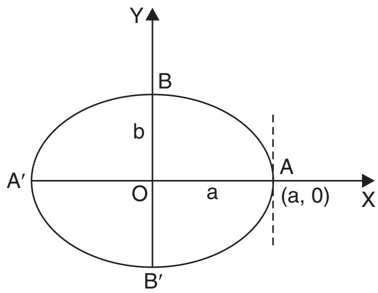
Sol. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or $$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \quad \therefore \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2) \qquad ...(1)$$

Since the ellipse is symmetrical about the x-axis

$\therefore$ the required volume = 2 × volume generated by revolving the area OABO about x-axis.

[We are multiplying by 2 because of symmetry about y-axis. See Note Cor. 2, Art. 2.]



Now for the area OABO, x varies from 0 to a.

$\therefore$ Required volume $= 2\int_0^a \pi y^2\, dx$

$$= 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\, dx \qquad [\because \text{ of } (1)]$$

$$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi \frac{b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right] = 2\pi \frac{b^2}{a^2} \cdot \frac{2a^3}{3} = \frac{4}{3}\pi a b^2.$$

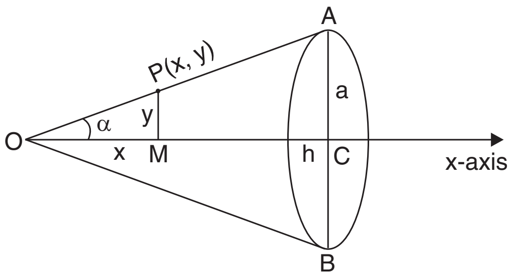
Note. Prolate and oblate spheroids. Defs. (*i*) The solid formed by the revolution of the ellipse about the **major axis** is called a **prolate spheroid** and (*ii*) the solid formed by the revolution of the ellipse about the **minor axis** is called an **oblate spheroid.**

Example 2. *Find the volume of a right circular cone of height h and radius a.*

Or

Find the volume of the right circular cone formed by revolution of a right angled triangle about a side which contains the right angle.

Sol. We know by example (*b*) Art. 1, that a right circular cone is generated by the revolution of a right angled triangle ΔOCA about is base OC (taken as x-axis here)



Let h be the height of the cone and a be the radius of its circular base.

In ΔOMP, $\tan\alpha = \frac{MP}{OM} = \frac{y}{x}$

$$\therefore \qquad y = x \tan\alpha \qquad ...(1)$$

Required volume of the cone $= \int_{x=0}^{h} \pi y^2 dx$

NOTES

Putting $y = x \tan \alpha$ from (1),

$$= \int_0^h \pi x^2 \tan^2 \alpha \, dx = \pi \tan^2 \alpha \int_0^h x^2 \, dx$$

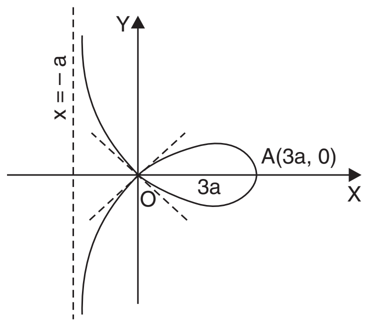
$$= \pi \tan^2 \alpha \left(\frac{x^3}{3} \right)_0^h = \frac{1}{3} \pi h^3 \tan^2 \alpha \qquad \text{...(2)}$$

Again in right angled ΔOCA, $\tan \alpha = \frac{AC}{OC} = \frac{a}{h}$ ...(3)

Putting this value of tan α from (3) in (2),

Required volume of the cone $= \frac{1}{3} \pi h^3 \cdot \frac{a^2}{h^2} = \frac{1}{3} \pi a^2 h.$

Example 3. *Find the volumes formed by the revolution of the loop of the curve $y^2 (a + x) = x^2 (3a - x)$, about the x-axis.*



Sol. The equation of the curve is

$$y^2(a + x) = x^2(3a - x) \qquad \text{...(1)}$$

(*i*) The curve is symmetrical about the *x*-axis.

(*ii*) The curves passes the origin and the tangents at the origin are $y = \pm \sqrt{3} \cdot x$

∴ Origin is a node.

(*iii*) The curve meets the *x*-axis in points (0, 0) and (3*a*, 0). It meets the *y*-axis at the origin only.

(*iv*) The curve has an asymptote $x + a = 0$, || to *y*-axis and there is no other asymptote.

(*v*) From (1), $y = x \cdot \sqrt{\frac{3a - x}{a + x}}$, when $x < -a$, or $x > 3a$, *y* is imaginary, ∴ the entire curve lies between the lines $x = -a$ and $x = 3a$.

Thus, the shape of curve as shown in the figure and for the upper half of its loop *x* varies from 0 to 3*a*.

Required volume formed by the revolution of loop

$$= \int_0^{3a} \pi y^2 \, dx = \pi \int_0^{3a} \frac{x^2 (3a - x)}{a + x} dx$$

$$= \pi \int_0^{3a} \frac{-x^3 + 3ax^2}{x + a} dx$$

$$= \pi \int_0^{3a} \left(-x^2 + 4ax - 4a^2 + \frac{4a^3}{x + a} \right) dx$$

[Dividing the Num. by the denominator]

$$= \pi \left[\frac{-x^3}{3} + 2ax^2 - 4a^2x + 4a^3 \log (x + a) \right]_0^{3a}$$

$$= \pi \left[-9a^3 + 18a^3 - 12a^3 + 4a^3 \log 4a - 4a^3 \log a \right]$$

$$= \pi a^3 (-3 + 4 \log 4) = \pi a^3 (8 \log 2 - 3).$$

Example 4. *A basin is formed by the revolution of the curve $x^3 = 64y$ $(y > 0)$ about the axis of y. If the depth of the basin is 8 inches, how many cubic inches of water will it hold ?*

Sol. The equation of the generating curve is $x^3 = 64y$ ...(1)

The curve is symmetrical in opposite quadrants.

The shape of the curve is as shown in the figure by thick lines.

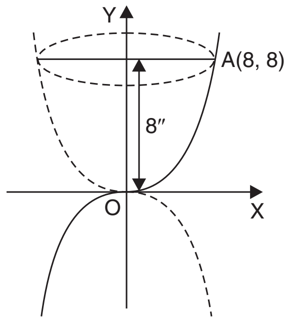
The height of basin is given to be 8 inches, so that when $y = 8$, from (1),

$$x^3 = 64 \times 8 \qquad \therefore \qquad x = 8.$$

Hence A(8, 8) is point of the curve (1) at a height of 8 inches.

Thus, the basin is formed by the revolution of the arc OA about the y-axis where A is (8, 8).

∴ Required volume



$$= \int_0^8 \pi x^2 dy = \int_0^8 \pi(64\,y)^{2/3}\, dy$$

$$= 16\pi \int_0^8 y^{2/3}\, dy$$

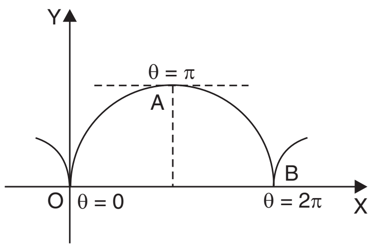
$$= 16\pi \,.\left[\frac{y^{5/3}}{\frac{5}{3}}\right]_0^8 = \frac{48}{5}\pi\ [(8)^{5/3} - 0]$$

$$= \frac{48\pi}{5} \times 32 = \frac{1536\pi}{5} \text{ cubic inches.}$$

Example 5. *Find the volume generated by revolving one arch of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about its base.*

Sol. The equations of the cycloid are $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$.

For the first half of the cycloid in the first quadrant, θ varies from 0 to π.



∴ By Cor 2, Art. 2 required volume

$$= 2\int_{\theta=0}^{\pi} \pi y^2\, dx = 2\int_0^{\pi} \pi y^2 \frac{dx}{d\theta}\, d\theta$$

$$= 2\pi \int_0^{\pi} a^2\,(1 - \cos\theta)^2 \,.\, a\,(1 - \cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (1 - \cos\theta)^3\, d\theta = 2\pi a^3 \int_0^{\pi} (2\sin^2\theta/2)^3\, d\theta \qquad \text{...(1)}$$

Put $\frac{\theta}{2} = t \quad \therefore \quad \theta = 2t \quad$ and $\quad d\theta = 2dt$

When $\theta = 0$, $t = 0$

When $\theta = \pi$, $t = \frac{\pi}{2}$

∴ Using (1)

Required volume $= 2\pi a^3 \int_0^{\pi/2} (2\sin^2 t)^3\, 2dt$

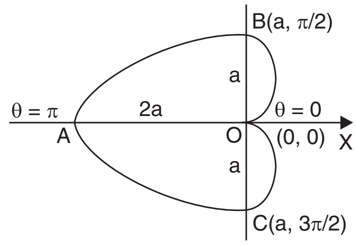
$$= 32\,\pi a^3 \int_0^{\pi/2} \sin^6 t\, dt = 32\,\pi a^3 \,.\, \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2} \,.\, \frac{\pi}{2} = 5\pi^2 a^3.$$

NOTES

Example 6. *Find the volume of the solid obtained by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line.*

Sol. The equation of the curve is $r = a(1 - \cos\theta)$.

The curve is symmetrical about the initial line and for its upper half θ varies from 0 to π.



$\therefore$ By Art. 3, required volume

$$= \int_0^{\pi} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} a^3 (1 - \cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi} \qquad \Bigg| \because \int (f(x))^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} \text{ if } n \neq -1$$

$$= \frac{2}{12}\pi a^3 \,[(1 - \cos\pi)^4 - (1 - \cos 0)^4]$$

$$= \frac{2}{12}\pi a^3 \,[2^4 - 0] = \frac{8}{3}\pi a^3.$$

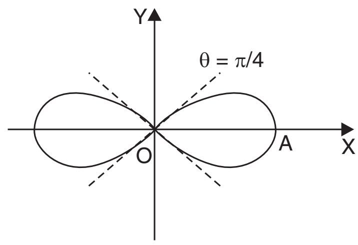
Example 7. *Find the volume of the solid obtained by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.*

Sol. The curve $r^2 = a^2 \cos 2\theta$ is symmetrical about the initial line.

The curve consists of two equal loops.

The required volume, therefore is twice the volume generated by the revolution of the half-loop OA about the initial line.

The equation of the curve is



$$r^2 = a^2 \cos 2\theta$$

or $$r^2 = a^2 (\cos^2\theta - \sin^2\theta)$$

or $$r^2 = a^2 \left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

i.e., $$r^4 = a^2 (x^2 - y^2)$$

or $$(x^2 + y^2)^2 = a^2 (x^2 - y^2)$$

or $$y^4 + (2x^2 + a^2)\, y^2 + (x^4 - a^2 x^2) = 0$$

It is a quadratic in y^2

Solving for y^2,

$$y^2 = \frac{-(2x^2 + a^2) \pm \sqrt{(2x^2 + a^2)^2 - 4(x^4 - a^2x^2)}}{2}$$

or $$y^2 = \tfrac{1}{2}\,[-(2x^2 + a^2) \pm \sqrt{8a^2x^2 + a^4}\,]$$

or $$y^2 = \tfrac{1}{2}\,[-(2x^2 + a^2) + \sqrt{8a^2x^2 + a^4}\,] \qquad ...(1)$$

Negative sign before the radical on the right is rejected, because otherwise y^2 is negative and hence y is imaginary.

For the half-loop OA, x varies from 0 to a.

$\therefore$ Required volume

$$= 2\int_0^a \pi y^2 \, dx$$

$$= \pi \int_0^a \left[-(2x^2 + a^2) + 2\sqrt{2}a\sqrt{x^2 + \frac{a^2}{8}} \right] dx \qquad \text{[From (1)]}$$

$$= \pi \left[-\frac{2}{3}x^3 - a^2x + 2a\sqrt{2} \left\{ \frac{x\sqrt{x^2 + \frac{a^2}{8}}}{2} + \frac{\left(\frac{a^2}{8}\right)}{2} \sinh^{-1} \frac{x}{\left(\frac{a}{2\sqrt{2}}\right)} \right\} \right]_0^a$$

$$= \pi \left[-\frac{2}{3}a^3 - a^3 + 2a\sqrt{2} \left\{ \frac{a \,.\, 3a}{2 \,.\, 2\sqrt{2}} + \frac{a^2}{16} \sinh^{-1} 2\sqrt{2} \right\} \right]$$

$$= \pi \left[-\frac{2}{3}a^3 - a^3 + \frac{3a^3}{2} + \frac{\sqrt{2}a^3}{8} \log(2\sqrt{2} + 3) \right] \qquad \left[\because \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) \right]$$

$$= \pi \left[-\frac{a^3}{6} + \frac{\sqrt{2}a^3}{8} \log(\sqrt{2} + 1)^2 \right]$$

$$= \frac{\pi a^3}{2} \left[\frac{-1}{3} + \frac{\sqrt{2}}{2} \log(\sqrt{2} + 1) \right]$$

$$= \frac{\pi a^3}{2} \left[\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1) - \frac{1}{3} \right].$$

EXERCISE 8.1

1. Find the volume generated by the revolution of an arc of the catenary $y = c \cosh \frac{x}{c}$ about the axis of x.

2. Find the volume generated by rotating about the y-axis the area bounded by the coordinate axes and the graph of the curve $y = \cos x$ from $x = 0$ to $x = \pi/2$.

$\left[\textbf{Hint.} \text{ Volume} = \int \pi x^2 \, dy = \int_0^{\pi/2} \pi x^2 \frac{dy}{dx} dx. \right]$

3. Let B be a number > 1. What is the volume of the solid generated by the area under the curve $y = e^{-x}$ between 1 and B (the axis of revolution being the x-axis) ? Does the volume approach a limit as B becomes large ? If so, what limit ?

4. The area of the parabola $y^2 = 4ax$ lying between the vertex and the latus rectum is revolved about the x-axis. Find the volume generated.

Or

Find the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

[**Hint.** We know that equation of latus rectum of the parabola $y^2 = 4ax$ is $x = a$.

$\therefore$ Required volume $= \int_0^a \pi y^2 \, dx = \int_0^a \pi \,.\, 4ax \, dx = 4a\pi \int_0^a x \, dx = 4a\pi \left(\frac{x^2}{2} \right)_0^a = 2\pi a^2.$]

5. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus formed.

[**Hint.** Tangent at the vertex is y-axis.]

6. Find the volume of a sphere of radius a.

[**Hint.** The equation of circle is $x^2 + y^2 = a^2$.

Volume of sphere $= 2\int_0^a \pi y^2\, dx = 2\int_0^a \pi(a^2 - x^2)\, dx.$]

7. (*i*) Find the volume of the oblate spheroid formed by revolving the ellipse about minor axis.

(*ii*) Find the volume of the solid generated by revolving the ellipse. $x^2 + 4y^2 = 4$ about y-axis.

8. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis, is a mean proportional (*i.e.* Geometric Mean) between those generated by the revolution of the ellipse and the auxiliary circle round the major axis.

[**Hint.** Equation of auxiliary circle (a circle drawn on major axis as diameter) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$.]

9. The loop of the curve $2ay^2 = x(x - a)^2$ revolves about x-axis, find the volume of the solid so generated.

10. Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2 = x^2(2a - x)(x - a)$ about x-axis.

11. (*a*) Find the volume formed by revolution of the loop of the curve $y^2 = \frac{x^2(a - x)}{a + x}$ about the x-axis.

(*b*) Find the volume of the solid produced by the revolution of the loop of the curve $y^2 = \frac{x^2(a + x)}{a - x}$ about the axis of x.

12. Find the volume of the solid generated by the revolution of the area between the curve $xy^2 = 4a^2(2a - x)$ and its asymptote about the asymptote.

13. (*a*) Find the volume of the solid formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, about its base.

[**Hint.** Base of the cycloid is x-axis.]

(*b*) Find the volume of the solid formed by revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$ about the tangent at the vertex.

[**Hint.** Tangent at the vertex is x-axis.]

14. Find the volume of the solid generated by revolving the curve $x^{2/3} + y^{2/3} = a^{2/3}$ (*i.e.* $x = a\cos^3 t$, $y = a\sin^3 t$) about x-axis.

15. Prove that the volume of the solid generated by the revolution of the tractrix $x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}$, $y = a\sin t$ about its asymptote equals half that of a sphere of radius a.

16. Find the volume of the solid generated by revolving the loop of the curve $x = t^2, y = t - \frac{1}{3}t^3$ about x-axis.

17. Find the volume of the solid generated by revolution of $r = 2a\cos\theta$ about the initial line.

18. The cardioid $r = a(1 + \cos\theta)$ revolves about the initial line. Find the volume of the solid generated.

19. The arc of the cardioid $r = a(1 + \cos\theta)$ specified by $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is rotated about the line $\theta = 0$, prove that the volume generated is $\frac{5}{2}\pi a^3$.

20. Show that the volume of the solid formed by the revolution of the curve $r = a + b\cos\theta$ $(a > b)$ about the initial line is $\frac{4}{3}\pi a(a^2 + b^2)$.

21. Find the volume of the solid generated by revolving the curve $r^2 = a^2 \cos 2\theta$ about the line $\theta = \pi/2$.

[**Hint.** $V = 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$, where $r = \sqrt{a^2 \cos 2\theta} = a\sqrt{1 - 2\sin^2\theta}$.

Now to integrate, put $\sqrt{2}\sin\theta = \sin\phi$.]

Answers

1. $\left[\frac{\pi c^2}{4}\left(2x + c\sinh\frac{2x}{c}\right)\right]_a^b$ **2.** $\pi(\pi - 2)$

3. $-\frac{\pi}{2}[e^{-2B} - e^{-2}], \frac{\pi}{2e^2}$ **4.** $2\pi a^3$ **5.** $\frac{4}{5}\pi a^3$

6. $\frac{4}{3}\pi a^3$ **7.** (*i*) $\frac{4}{3}\pi a^2 b$ (*ii*) $\frac{16\pi}{3}$

9. $\frac{\pi a^3}{24}$ **10.** $\frac{23\pi a^3}{60}$

11. (*a*) $2\pi a^3\left[\log 2 - \frac{2}{3}\right]$ (*b*) $2\pi a^3\left[\log 2 - \frac{2}{3}\right]$

12. $4\pi^2 a^3$ **13.** (*a*) $5\pi^2 a^3$ (*b*) $\pi^2 a^3$

14. $\frac{32}{105}\pi a^3$ **16.** $\frac{3\pi}{4}$ **17.** $\frac{4}{3}\pi a^3$

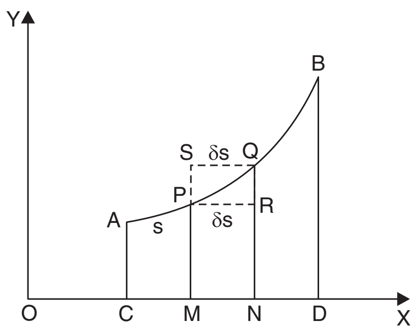
18. $\frac{8}{3}\pi a^3$ **21.** $\frac{\pi^2 a^3}{4\sqrt{2}}$.

SURFACES OF THE SOLIDS OF REVOLUTION

The area of the surface of the solid generated by the revolution about x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates

$x = a$, $x = b$, is $\int_{x=a}^{x=b} \mathbf{2\pi y\, ds}$

where s is the length of the arc of curve measured from a fixed point on it to any point (x, y).



Let AB be the curve $y = f(x)$ and CA, BD the ordinates $x = a$, $x = b$. Let P(x, y) be *any* point on the curve and let the arc AP be s. Draw PM ⊥ OX.

If S denotes the curved surface of the solid generated by the revolution of the area ACMP about the x-axis, then clearly S is a function of s.

NOTES

Let Q($x + \delta x, y + \delta y$) be a point on the curve in the neighbourhood of P. Draw QN ⊥ OX, and let the arc AQ be $s + \delta s$, so that arc PQ = δs.

∴ The curved surface of the solid generated by the revolution of the area PMNQ about the x-axis is δS. Draw PR and QS parallel to x-axis and each equal in length to the arc PQ *i.e.*, δs.

Then it may be taken as an axiom that the curved surface of the solid generated by the revolution of the area PMNQ lies between the curved surfaces of the right circular cylinders whose base radii are MP, NQ and heights are PR, QS.

∴ δS lies between $2\pi y\, \delta s$ and $2\pi(y + \delta y)\, \delta s$

or $$2\pi y\, \delta s < \delta S < 2\pi\, (y + \delta y)\, \delta s \quad \text{or} \quad 2\pi y < \frac{\delta S}{\delta s} < 2\pi\, (y + \delta y).$$

Since $y = f(x)$ is continuous, ∴ as $\delta x \to 0$, δy also $\to 0$.

∴ Proceeding to the limit as $\delta x \to 0$ (∴ $\delta y \to 0$),

$$\frac{dS}{ds} = 2\pi y \qquad \ldots(1)$$

$$\therefore \quad \int_{x=a}^{x=b} 2\pi y\, ds = \int_{x=a}^{x=b} \frac{dS}{ds}\, ds = [S]_{x=a}^{x=b}$$

= (value of S when $x = b$) – (value of S when $x = a$)

= curved surface of the solid generated by the revolution of the area ABCD – 0.

∴ Surface area of the solid generated by the revolution of the area ACDB

$$= \int_{x=a}^{x=b} 2\pi y\, ds.$$

Note 1. The above result has been proved for the case when the ordinate increases continuously. It is equally true in case, where the ordinate decreases continuously. It is also assumed that the curve does not cross the x-axis *i.e.*, the axis of revolution.

Cor. From (1), $S = \int 2\pi y\, ds$.

Practical forms of the surface formula

(*i*) **Cartesian form [for the curve y = f(x)]**

$$S = \int 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

(*ii*) **Parametric form [for the curve x = f(t), y = F(t)]**

$$S = \int 2\pi y \frac{ds}{dt}\, dt, \text{ where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

(*iii*) **Polar form [for the curve r = f(θ)]**

$$S = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where}$$

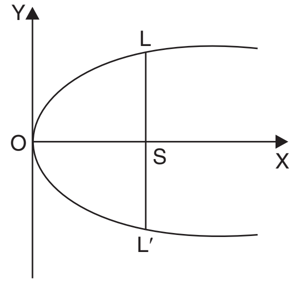
$$y = r \sin\theta \text{ and } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Note 2. *Interchanging* x and y in the above formula, we see that the curved surface of the solid generated by the revolution about y-axis of the area bounded by the curve $x = f(y)$, y-axis, and abscissae $y = c$, $y = d$ is $\int_{y=c}^{y=d} 2\pi x\, ds$.

NOTES

Example 1. *Find the area of the surface formed by the revolution of $y^2 = 4ax$, about the x-axis, by the arc from the vertex to one end of latus rectum.*

Sol. The equation of the parabola is $y^2 = 4ax$.



Differentiating, we get $2y\dfrac{dy}{dx} = 4a$ or $\dfrac{dy}{dx} = \dfrac{2a}{y}$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4a^2}{y^2}}$$

$$= \sqrt{1+\frac{4a^2}{4ax}} \qquad (\because \quad y^2 = 4ax)$$

$$= \sqrt{\frac{x+a}{x}}$$

For the arc from the vertex O to L, the end of the latus rectum x varies from 0 to a.

$$\therefore \quad \text{Required surface} = \int_0^a 2\pi y \frac{ds}{dx}\, dx \qquad [\textit{Practical form (i)}]$$

$$= 2\pi \int_0^a \sqrt{4ax} \,.\, \sqrt{\frac{x+a}{x}}\, dx = 4\pi\sqrt{a}\int_0^a (x+a)^{1/2}\, dx$$

$$= 4\pi\sqrt{a}\left|\frac{(x+a)^{3/2}}{3/2}\right|_0^a = \frac{8\pi}{3}\sqrt{a}\ [(2a)^{3/2} - a^{3/2}]$$

$$= \frac{8}{3}\pi a^2 (2\sqrt{2} - 1).$$

Example 2. *Prove that the surface generated by revolution of the tractrix $x = a \cos t + \dfrac{a}{2} \log \tan^2 \dfrac{t}{2}$, $y = a \sin t$ about its asymptote is equal to the surface of a sphere of radius a.*

Sol. The equations of the curve are

$$\left.\begin{aligned} x &= a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2} = a\cos t + a\log\tan\frac{t}{2} \\ y &= a\sin t. \end{aligned}\right\} \qquad ...(1)$$

The curve (1) is symmetrical about both the axes and its asymptote is $y = 0$, *i.e.*, x-axis.

From (1),
$$\frac{dx}{dt} = -a\sin t + a\,.\,\frac{1}{\tan\frac{t}{2}}\sec^2\frac{t}{2}\,.\,\frac{1}{2} = -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -a\sin t + \frac{a}{\sin t} = a\frac{(1-\sin^2 t)}{\sin t} = a\frac{\cos^2 t}{\sin t}$$

and
$$\frac{dy}{dt} = a\cos t$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t}$$

$$= \sqrt{\frac{a^2\cos^2 t\,(\cos^2 t + \sin^2 t)}{\sin^2 t}} = \frac{a\cos t}{\sin t}$$

For the curve in the second quadrant t varies from 0 to $\pi/2$.

$\therefore$ Required surface $= 2 \int_0^{\pi/2} 2\pi y \dfrac{ds}{dt}\, dt$ [*Practical Form* (*ii*)]

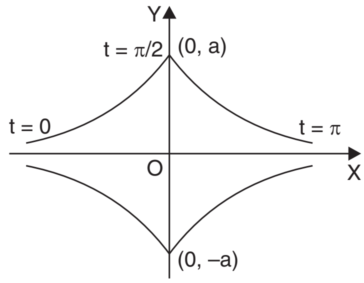
$= 4\pi \int_0^{\pi/2} a \sin t \,.\dfrac{a \cos t}{\sin t}\, dt$

$= 4\pi a^2 \int_0^{\pi/2} \cos t\, dt$

$= 4\pi a^2 \left[\sin t\right]_0^{\pi/2}$

$= 4\pi a^2 (1 - 0) = 4\pi a^2$

= Surface of a sphere of radius a.



Example 3. *The arc of the cardioid* $r = a(1 + \cos\theta)$ *included between* $-\pi/2 \le \theta \le \pi/2$ *is rotated about line* $\theta = \pi/2$. *Find the area of the surface generated.*

Sol. Equation of cardioid is $r = a(1 + \cos\theta)$

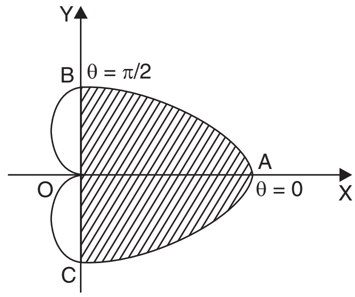
The area OCAB (shown shaded in the figure) revolves about the line $\theta = \pi/2$ *i.e.,* y-axis. Also the curve is symmetrical about the initial line or x- axis.

$\therefore$ Required surface area

= 2 × surface generated by revolution of arc AB about y-axis.

$= 2 \int_0^{\pi/2} 2\pi x \dfrac{ds}{d\theta}\, d\theta$ ...(2)

(See Note 2, Art. 4)



From (1), $\dfrac{dr}{d\theta} = -a \sin\theta$

$\therefore$ $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = \sqrt{a^2 (1 + \cos\theta)^2 + a^2 \sin^2\theta}$

$= a\sqrt{2(1 + \cos\theta)} = a \,.\, 2\cos\theta/2 = 2a\cos\theta/2$ and $x = r\cos\theta$.

Now from (2), we have required surface area $= 4\pi \int_0^{\pi/2} x \dfrac{ds}{d\theta}\, d\theta$

$= 4\pi \int_0^{\pi/2} r\cos\theta \,.\, 2a\cos\theta/2\, d\theta$

$= 4\pi \int_0^{\pi/2} a(1 + \cos\theta) \,.\, \cos\theta \,.\, 2a\cos\theta/2\, d\theta$

$= 8\pi a^2 \int_0^{\pi/2} (1 + \cos\theta)\cos\theta \,.\, \cos\theta/2\, d\theta$

$= 8\pi a^2 \int_0^{\pi/2} (2 - 2\sin^2\theta/2)(1 - 2\sin^2\theta/2) \,.\, \cos\theta/2\, d\theta$

$\left(\because\ \cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}\right)$

$= 16\pi a^2 \int_0^{\pi/2} (1 - \sin^2\theta/2)(1 - 2\sin^2\theta/2) \,.\, \cos\theta/2\, d\theta$

$$= 16\pi a^2 \int_0^{\pi/2} [1 - 3\sin^2\theta/2 + 2\sin^4\theta/2] \cos\theta/2 \, d\theta$$

$$\left[\text{Put } \sin\frac{\theta}{2} = t, \text{ so that } \frac{1}{2}\cos\frac{\theta}{2}\, d\theta = dt \text{ and} \right.$$

$$\left. \text{when } \theta = 0, t = 0 \text{ ; when } \theta = \frac{\pi}{2} ; t = \frac{1}{\sqrt{2}} \right]$$

$$= 16\pi a^2 \int_0^{1/\sqrt{2}} (1 - 3t^2 + 2t^4)\, 2dt = 32\pi a^2 \left[t - t^3 + \frac{2t^5}{5} \right]_0^{1/\sqrt{2}}$$

$$= 32\pi a^2 \left[\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{2}{5} \cdot \frac{1}{4\sqrt{2}} \right]$$

$$= 32\pi a^2 \left[\frac{20 - 10 + 2}{20\sqrt{2}} \right] = 32\pi a^2 \times \frac{12}{20\sqrt{2}} = \frac{96\pi a^2}{5\sqrt{2}}.$$

EXERCISE 8.2

1. Find the surface generated by the revolution of an arc of the catenary $y = c \cosh \frac{x}{c}$ about the axis of x.
2. Find the area of the surface formed by revolution of $y^2 = 4ax$ about y-axis by the arc from the vertex to $x = \frac{a}{4}$.
3. Find the surface of a sphere of radius a.

   Or

   The circle $x^2 + y^2 = a^2$ is revolved about x-axis. Find the area of the sphere generated.
4. Find the area of the surface generated by rotating about x-axis the arc of the curve $y = x^3$ between $x = 0$ and $x = 1$.
5. Find the surface of the right circular cone formed by the revolution of a right-angled triangle about a side which contains the right angle.
6. Find the surface of the solid generated by the revolution of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ or $x = a\cos^3 t$, $y = a\sin^3 t$ about the x-axis.
7. Find the surface area of the solid generated by revolving the cycloid :

   (*a*) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

   (*b*) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex.

   [**Hint.** Tangent at the vertex is x-axis.]

   (*c*) Find the area of the curved surface generated by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.
8. Find the surface of the solid generated by revolving the loop of the curve $x = t^2$, $y = t - \frac{1}{3} t^3$ about x-axis.
9. Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.
10. (*a*) The curve $r = a(1 + \cos\theta)$ revolves about the initial line. Find the surface of the figure so formed.

    (*b*) Find the surface of the solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.

NOTES

11. Find the surface of the solid generated by the revolution of the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.

12. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.

13. Prove that the surface of the solid obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about x-axis is $2\pi ab\left[\sqrt{1-e^2} + \frac{1}{e}\sin^{-1} e\right]$, where e is the eccentricity of the ellipse.

14. Show that the surface of the spherical zone contained between two parallel planes is $2\pi ah$, where a is the radius of the sphere and h is the distance between parallel planes.

15. A quadrant of a circle of radius a revolves round its chord. Find the area of the surface of the spindle generated.

Answers

1. $\pi c\left[b - a + \frac{c}{2}\sinh\frac{2b}{c} - \frac{c}{2}\sinh\frac{2a}{c}\right]$ **2.** $\frac{\pi a^2}{16}\left[3\sqrt{5} - 8.\log\sqrt{5} + \frac{1}{2}\right]$

3. $4\pi a^2$ **4.** $\frac{\pi}{27}(10\sqrt{10} - 1)$

5. πrl **6.** $\frac{12}{5}\pi a^2$ **7.** (*a*) $\frac{64}{3}\pi a^2$ (*b*) $\frac{32}{3}\pi a^2$ (*c*) $\frac{32}{3}\pi a^2$

8. 3π **9.** $4\pi a^2$ **10.** (*a*) $\frac{32}{5}\pi a^2$ (*b*) $\frac{32}{5}\pi a^2$

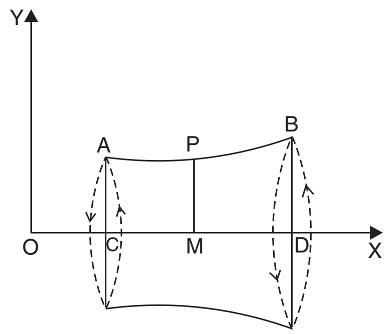
11. $4\pi a^2\left(1 - \frac{1}{\sqrt{2}}\right)$ **12.** $\pi a^2\,[3\sqrt{2} - \log(\sqrt{2} + 1)]$ **15.** $2\sqrt{2}\,\pi a^2\left(1 - \frac{\pi}{4}\right)$.

REVOLUTION ABOUT ANY AXIS

The **volume** *of the solid generated by the revolution, about any axis CD, of the area bounded by the curve AB, the axis CD and the perpendiculars AC, BD, on the axis is*

$$\int_{OC}^{OD} \pi\,(PM)^2\, d(OM)$$

where O is a fixed point on the axis CD, and PM is perpendicular from any point P of the curve AB on the axis CD.



Take the fixed point O on CD as origin ; and OCD the axis of revolution as the x-axis, and OY, perpendicular to it as the y-axis.

Let the co-ordinates of P be (x, y), so that OM $= x$, PM $= y$.

If OC $= a$ and OD $= b$, then the required volume

$$= \int_a^b \pi y^2\,dx = \int_{OC}^{OD} \pi(PM)^2\, d(OM).$$

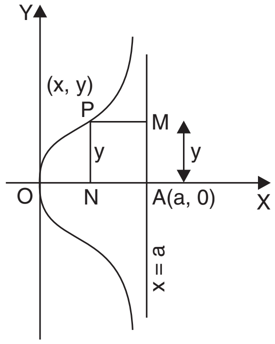
Example 1. *Show that the volume of the solid generated by the revolution of the curve $(a - x)y^2 = a^2x$ about its asymptote is $\frac{1}{2}\pi^2 a^3$.*

Sol. The equation of the curve is $(a - x)y^2 = a^2x$ ...(1)

The curve is symmetrical about the x-axis and by equating to zero the co-efficient of the highest power of y the asymptote parallel to y-axis is

$$a - x = 0 \text{ or } x = a$$

Let P(x, y) be any point on the curve and PM, the perpendicular on the asymptote (the axis of revolution), the PM = $a - x$, and AM = y, ON = x ; A being the point of intersection of the asymptote with the x-axis.



$\therefore$ Required volume $= 2 \int \pi \, (\text{PM})^2 \, d(\text{AM})$

$$= 2 \int_0^{\infty} \pi (a - x)^2 \, dy \quad \left[\because \text{ for upper half of curve } y \text{ varies from 0 to } \infty\right] \quad \ldots(2)$$

$$= 2\pi \int_0^{\infty} \left[a - \frac{ay^2}{a^2 + y^2}\right]^2 dy$$

$$\left[\because \text{ From (1), } x(a^2 + y^2) = ay^2 \text{ so that } x = \frac{ay^2}{a^2 + y^2}\right]$$

$$= 2\pi \int_0^{\infty} \frac{a^6 dy}{(y^2 + a^2)^2} \quad \left\{\begin{array}{l}\text{Put } y = a \tan\theta \;\therefore dy = a \sec^2\theta \, d\theta \\ \text{when } y = 0, \theta = 0 \text{ and} \\ \text{when } y = \infty, \theta = \pi/2.\end{array}\right\}$$

$$= 2\pi \int_0^{\pi/2} \frac{a^6 \cdot a \sec^2\theta \, d\theta}{a^4 \sec^4\theta}$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2\theta \, d\theta = 2\pi a^3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2}\pi^2 a^3.$$

Note. From (2), we can also proceed as follows :

We have AM $= y = \dfrac{a\sqrt{x}}{\sqrt{a - x}}$

$$\therefore \quad d(\text{AM}) = dy = a. \frac{\sqrt{a - x} \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot \frac{1}{2\sqrt{a - x}}(-1)}{a - x} dx$$

$$= \frac{a^2 \, dx}{2\sqrt{x} \cdot (a - x)^{3/2}}$$

$\therefore$ From (2), the required volume

$$= 2 \int \pi \, (\text{PM})^2 \, d(\text{AM}) = 2\pi \int_0^a (a - x)^2 \, dy$$

$$= 2\pi \int_0^a (a - x)^2 \cdot \frac{a^2 \, dx}{2\sqrt{x} \cdot (a - x)^{3/2}}$$

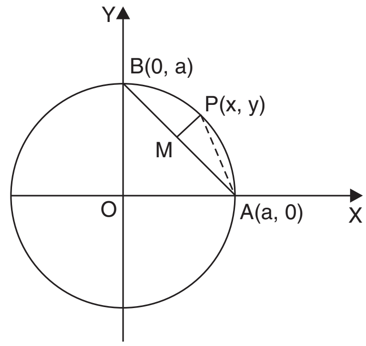
$$= \pi a^2 \int_0^a \frac{\sqrt{a - x}}{\sqrt{x}} dx \quad \left[\begin{array}{l}\text{Put } x = a \sin^2\theta \\ \therefore dx = 2a \sin\theta \cos\theta \, d\theta \\ \text{When } x = 0, \theta = 0, \text{ and when } x = a, \theta = \frac{\pi}{2}\end{array}\right]$$

$$= \pi a^2 \int_0^{\pi/2} \frac{\sqrt{a \cos^2\theta}}{\sqrt{a \sin^2\theta}} \cdot 2a \sin\theta \cos\theta \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2\theta \, d\theta = 2\pi a^3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2}\pi^2 a^3.$$

Example 2. *A quadrant of a circle of radius a, revolves about its chord. Show that the volume of the spindle generated is* $\frac{\pi}{6\sqrt{2}}(10-3\pi)a^3$.

NOTES



Sol. Let the equation of the generating circle be $x^2 + y^2 = a^2$ ...(1)

If A and B be the extremities of the arc (in the positive quadrant), then the equation of line AB is

$$y - 0 = \frac{0-a}{a-0}(x-a)$$

or $$x + y - a = 0.$$

If P(x, y) be any point on the arc AB. Draw PM ⊥ on chord AB and join AP.

Then $$\text{PM} = \frac{x+y-a}{\sqrt{2}} = \frac{x-a+\sqrt{a^2-x^2}}{\sqrt{2}} \quad \text{[From (1)]}$$

and $$\text{PM}^2 = \frac{x^2+a^2-2ax+a^2-x^2+2(x-a)\sqrt{a^2-x^2}}{2}$$

$$= (a-x)\,(a-\sqrt{a^2-x^2})$$

Now $$\text{AM}^2 = \text{AP}^2 - \text{PM}^2$$

$$= (x-a)^2 + y^2 - \frac{(x+y-a)^2}{2}$$

$$= \frac{1}{2}[2(x-a)^2 + 2y^2 - (x-a)^2 - y^2 - 2y(x-a)]$$

$$= \frac{1}{2}[(x-a)^2 + y^2 - 2y(x-a)] = \frac{1}{2}(x-a-y)^2$$

∴ $$\text{AM} = \frac{1}{\sqrt{2}}(x-y-a) = \frac{1}{\sqrt{2}}(x-a-\sqrt{a^2-x^2}) \quad \text{[From (1)]}$$

∴ $$d(\text{AM}) = \frac{1}{\sqrt{2}}\left[1+\frac{x}{\sqrt{a^2-x^2}}\right]dx = \frac{\sqrt{a^2-x^2}+x}{\sqrt{2}\sqrt{a^2-x^2}}\,dx$$

Also for the arc AB, x varies from 0 to a.

∴ Required volume $= \int \pi\,(\text{PM})^2\, d(\text{AM})$

$$= \int_0^a \pi(a-x)(a-\sqrt{a^2-x^2})\,.\,\frac{\sqrt{a^2-x^2}+x}{\sqrt{2}\sqrt{a^2-x^2}}\,dx$$

$$\left[\begin{array}{l}\text{Put } x = a\sin\theta, \text{ so that } dx = a\cos\theta\,d\theta, \\ \text{and when } x = 0, \theta = 0, \text{ when } x = a, \theta = \pi/2\end{array}\right]$$

$$= \int_0^{\pi/2} \pi\,.\,a(1-\sin\theta)\,.\,a(1-\cos\theta)\,.\,\frac{a\cos\theta + a\sin\theta}{\sqrt{2}\,.\,a\cos\theta}\,.\,a\cos\theta\,d\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\int_0^{\pi/2}(1-\sin\theta)(1-\cos\theta)(\sin\theta+\cos\theta)\,d\theta$$

$$= \frac{\pi a^3}{\sqrt{2}}\int_0^{\pi/2}(\sin\theta+\cos\theta-2\sin\theta\cos\theta-1+\sin^2\theta\cos\theta+\cos^2\theta\sin\theta)\,d\theta$$

NOTES

$$= \frac{\pi a^3}{\sqrt{2}} \left[-\cos\theta + \sin\theta + \cos^2\theta - \theta + \frac{1}{3}\sin^3\theta - \frac{1}{3}\cos^3\theta \right]_0^{\pi/2}$$

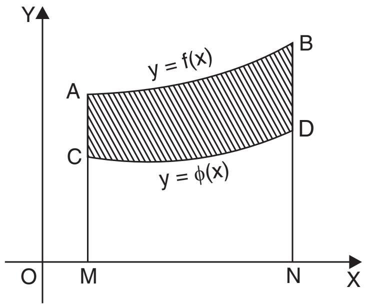
$$= \frac{\pi a^3}{\sqrt{2}} \left[\left(-0 + 1 + 0 - \frac{\pi}{2} + \frac{1}{3} - 0 \right) - \left(-1 + 0 + 1 - 0 + 0 - \frac{1}{3} \right) \right]$$

$$= \frac{\pi a^3}{\sqrt{2}} \left(\frac{5}{3} - \frac{\pi}{2} \right) = \frac{\pi a^3}{6\sqrt{2}} (10 - 3\pi).$$

VOLUME BETWEEN TWO SOLIDS

The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curves $y = f(x)$, $y = \phi(x)$ and the ordinates $x = a$, $x = b$ is

$$\int_a^b \pi[(\text{y of upper curve})^2 - (\text{y of lower curve})^2]\, dx.$$



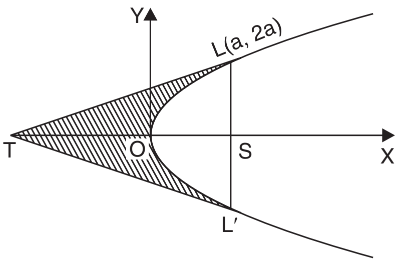
Let AB be the curve $y = f(x)$ and CD be the curve $y = \phi(x)$ both between the ordinates MCA ($x = a$) and NDB ($x = b$).

∴ Volume of the solid generated by the revolution about x-axis, of the shaded area ACDB.

= volume of the solid generated by revolving the area AMNB about x-axis – volume of the solid generated by revolution about x-axis, of the area CMND.

$$= \int_a^b \pi\, [f(x)]^2\, dx - \int_a^b \pi[\phi(x)]^2\, dx.$$

Example 1. *The figure bounded by a parabola and the tangents at the extremities of its latus rectum revolves about the axis of the parabola. Find the volume of the solid thus obtained.*



Sol. Let the equation of the parabola be $y^2 = 4ax$, and let LSL′ be its latus rectum and LT, L′T the tangents at L and L′. The equation of the tangent at L $(a, 2a)$ is $y.2a = 2a(x + a)$ or $y = x + a$.

This meets x-axis in T$(- a, 0)$. The curve and the two tangents TL, TL′, are symmetrical about x-axis.

∴ Required volume of the solid formed by revolving the area TOL about the axis of parabola (*i.e.*, x-axis) = volume of the solid generated by revolving the area TSL about x-axis –volume of the solid generated by the revolution about x-axis of the area OLS.

$$= \int_{-a}^{a} \pi y^2\, dx - \int_0^a \pi y^2\, dx$$

(for tangent) (for curve)

$$= \int_{-a}^{a} \pi(x + a)^2\, dx - \int_0^a \pi \,.\, 4ax\, dx$$

$$= \pi \left[\frac{(x + a)^3}{3} \right]_{-a}^{a} - 2a\pi \left[x^2 \right]_0^a = \frac{\pi}{3} (8a^3 - 0) - 2a\pi \,.\, a^2 = \frac{2}{3} \pi a^3.$$

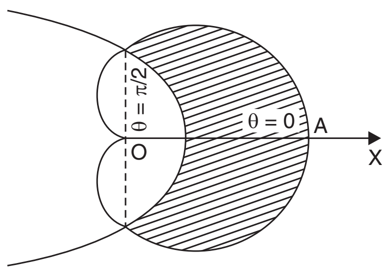
NOTES

Example 2. *Show that if the area lying within the cardioid $r = 2a(1 + \cos\theta)$ and without the parabola $r(1 + \cos\theta) = 2a$, revolves about the initial line, the volume generated is $18\pi a^3$.*

Sol. The equation of the cardioid is $r = 2a(1 + \cos\theta)$ ...(1)

and the equation of the parabola is $r = \dfrac{2a}{1+\cos\theta}$ ...(2)

Both the curves are symmetrical about the initial line.



The upper half of the shaded area revolves about the initial line and it will generate the required volume.

The two curves intersect where solving (1) and (2). (Eliminating r) [*i.e.* Putting the value of r from (1) in (2)]

$$2a(1 + \cos\theta) = \frac{2a}{1+\cos\theta}$$

or $(1 + \cos\theta)^2 = 1$ or $1 + 2\cos\theta + \cos^2\theta = 1$

or $\cos\theta(2 + \cos\theta) = 0$ $\therefore$ $\cos\theta = 0$ or -2

But $\cos\theta \neq -2$, [$\because$ $\cos\theta$ can never be numerically > 1]

$\therefore$ $\cos\theta = 0$ or $\theta = \pm\pi/2$.

For the upper half, θ varies from 0 to $\pi/2$.

$\therefore$ Required volume

$$= \int_0^{\pi/2} \frac{2\pi}{3}\,[(r \text{ of outer curve})^3 - (r \text{ of inner curve})^3] \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi/2}\left[[2a(1+\cos\theta)]^3 - \left(\frac{2a}{1+\cos\theta}\right)^3\right]\sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\,.\,8a^3\left[-\frac{(1+\cos\theta)^4}{4} + \frac{(1+\cos\theta)^{-2}}{-2}\right]_0^{\pi/2}$$

$$= \frac{16\pi a^3}{3}\left[-\left(\frac{1-16}{4}\right) - \frac{1}{2}\left(1-\frac{1}{4}\right)\right]$$

$$= \frac{16\pi a^3}{3}\left(\frac{15}{4} - \frac{3}{8}\right) = \frac{16\pi a^3}{3} \times \frac{27}{8} = 18\,\pi a^3.$$

REVOLUTION ABOUT ANY AXIS

The **surface** *area of the solid generated by the revolution about any axis CD of the arc AB is*

$$\int_{\mathbf{x}=\mathbf{OC}}^{\mathbf{x}=\mathbf{OD}} 2\pi\,(\mathbf{PM})\,.\,\mathbf{ds}$$

where PM, AC, BD are perpendiculars from any point P, A and B of the arc AB on CD and arc PA = s and O is any fixed point on the axis CD.

Take the fixed point O as origin, the axis of revolution OCD as the x-axis and OY perpendicular to it as y-axis (*Refer to figure of Art. 5,*).

NOTES

Let the co-ordinates of P be (x, y) so the OM = x and MP = $\overline{y}$.

Then the curved surface

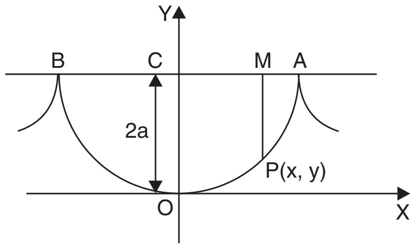
$$= \int_{x=\text{OC}}^{x=\text{OD}} 2\pi\, y\, ds = \int_{x=\text{OC}}^{x=\text{OD}} 2\pi\,(\text{PM})\, ds.$$

Example 1. *Find the area of the curved surface generated by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.*

Sol. The equations of the cycloid are $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

The cycloid is symmetrical about the y-axis.

For the arc OPA of the cycloid θ varies from 0 to π. The cycloid revolves about the base BA, which is a line parallel to x-axis and at a distance $2a$ from it.



Now if P(x, y) be any point on the cycloid and PM $\perp$ AB (axis of revolution), then PM $= 2a - y$.

$\therefore$ Required surface

$$= 2 \times \text{surface generated by the arc OPA}$$

$$= 2\int_0^{\pi} 2\pi\,(\text{PM})\,\frac{ds}{d\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi} (2a - y)\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\pi \int_0^{\pi} (2a - a + a\cos\theta)\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} (1 + \cos\theta)\sqrt{2(1+\cos\theta)}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} (2\cos^2\theta/2)(2\cos\theta/2)\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \cos^3 t \,.\, 2\, dt, \text{ where } t = \theta/2$$

$$= 32\,\pi a^2 \,.\, \frac{2}{3} = \frac{64\pi a^2}{3}.$$

EXERCISE 8.3

1. (*a*) Show that the volume of the solid formed by the revolution of the cissoid $y^2(2a - x) = x^3$ about its asymptote is $2\pi^2 a^3$.

   (*b*) Find the volume of the solid generated by the arc of the cissoid $x = 2a\sin^2 t$, $y = \dfrac{2a\sin^3 t}{\cos t}$ about its asymptote.

2. Find the volume of the solid of revolution obtained by rotating the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about x-axis.
3. Find the area of the surface generated if an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ revolves about the line $y = 2a$.

Answers

1. (*a*) $2\pi^2a^3$ (*b*) $2\pi^2a^3$ **2.** $\frac{5}{28}\pi$. **3.** $\frac{32}{3}\pi a^2$.

NOTES

THEOREMS OF PAPPUS AND GULDIN

(*i*) If a closed plane curve revolves about an axis in its plane, which does not intersect it, then the volume of the solid generated by the area enclosed by the curve is equal to the product of the area enclosed by the curve and the length of the path described by the centre of gravtiy of this area.

(*ii*) If an arc of a plane curve revolves about an axis in its plane, which does not intersect it, then the area of the curved surface generated by the arc is equal to the product of the length of the arc and the length of the path described by the centre of gravity of this arc.

Proof. (*i*) Let A be the area of the closed curved and θ, the angle through which it revolves

Take the axis of rotation as the *x*-axis and a line perpendicular to it as *y*-axis. Consider an element of area δA at any pont P(*x, y*) which after the rotation through an angle θ radians takes the position P.

The length of the arc described by O is *y*θ. (I = *r*θ) Hence the volume δV of the solid generated by the element = *y*θ . δA.

The volume generated by the whole area

$$= \Sigma y\theta \,.\, \delta A = \theta \Sigma y \delta A = \theta \int y\,dA$$

the integration being taken over the whole area.

If *y* be the ordinate of the centre of gravity of the area, then

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int y\,dA}{A}$$

$$\therefore \quad \int y\,dA = A\bar{y}$$

Substituting in (1), the volume generated by the whole area

$$= \theta \,.\, Ay = A(\bar{y}\theta)$$

= area of the curve × length of path described by the C.G of the area

(*i*) Let s be the length of the arc of the curve and δ*s*, an element of the arc at any point Q(*x*′, *y*′) on the curve

Then the length of the arc described by Q in a rotation through an angle θ is *y*′θ. Therefore, the area of the surface generated by the element ds in *y*′θ.δ*s*

Hence the surface area generated byt he whole perimeter

$$= \Sigma y'\theta.\delta s = \theta \int y'ds \qquad \ldots(2)$$

NOTES

If $\overline{y}$ is the ordinate of the centre of gravity of the arc, then

$$\overline{y} = \frac{\int y'ds}{\int ds} = \frac{\int y'ds}{s} \qquad \text{[From Statics]}$$

$$\therefore \qquad \int y'ds = s\overline{y}$$

Substituting in (2), the required surface area

$= \theta.s\overline{y} = s(\overline{y}\theta)$

= the length of the arc × the length of the path described by the C.G. of the arc.

Note:

1. In (*i*) above, the axis of revolution should not intersect the curve, it may however, touch it. In (*ii*) above, the revolving arc should not cross the axis of revolution. but it may be terminated by it, or it may touch it.
2. If the curve revolves through four rt. $\angle s$, $\theta = 2\pi$

Example 1. *Find the volume and the surface of the anchorring formed by the revolution of a circcle of raduis about a line in its place at a distane d from the centre, (d > a).*

Sol. Area of the circle = πa^2

Circumference of circle = $2\pi a$

The centre is at a distance d from the axis of revolution

∴ Length of the path described by C.G. = 2pd (Here $\theta = 2\pi$)

Volume of the anchor-ring

$$= \pi a^2 \times 2\pi d = 2\pi^2 a^2 d$$

$$\text{Surface area} = 2\pi a \times 2\pi d = 4\pi^2 ad$$

Example 2. *The coordinates of the vertices of a rectangular lamna ABCD are A(2, 2), B(6, 2), C(6, 4) and D(2, 4). Using the theorem of Pappus, find the surface of the solid obtained by revolving the rectangle about the line x = 9.*

Sol. AB = 4, BC = 2

∴ Perimeter of rectangle = 2(4 + 2) = 12

C.G. of lamina s G(4, 3) the mid point of diagonals

Distance of G from y-axis = 4

Distance of axis from y-axis = 9

∴ Distance of G from y-axis = 9 – 4 = 5

Distance moved by G in one revolution

$$= 5 \times 2\pi = 10\pi$$

∴ Required surface of the solid = $12 \times 10\pi = 120\pi$

Example 3. *A square of side a revolves about a line through a corner and perpendicular to the diagonal through that corner. Find the volume and area of the surface of the solid generated.*

Sol. Area of square = a^2

Perimeter of square = $4a$

Let G be the C.G. of the square. Then AM = GM = $a/2$

$\therefore$ AG = distance of G from axis of revolution

NOTES

$$= \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}}$$

Distance moved by G in one revolution

$$\frac{a}{\sqrt{2}} \times 2\pi = \sqrt{2}a\pi$$

$\therefore$ Required volume

$$a^2 \times \sqrt{2}a\pi = \sqrt{2}\pi a^3$$

Area of surface $4a \times \sqrt{2}a\pi = 4\sqrt{2}\pi a^2$

Example 4. *The ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *revolves about a tangent at an end of (i) the major axis (ii) the minor axis. Show that the volumes generated are* $2\pi^2 a^2 b$ *and* $2\pi^2 ab^2$ *respectively.*

Sol. Area of ellipse = πab

(*i*) Distance of G, the C.G. of ellipse (i.e., centre) from the tangent at an end of the major axis = a. Distance moved by G in one revolution = $2\pi a$

$\therefore$ Required volume = $\pi ab \times 2\pi a = 2\pi^2 ab$

(*ii*) Distance of G from the tangent at an end of the minor axis = b. Distance moved by G in one revolution = $2\pi b$

$\therefore$ Required volume = $\pi ab \times 2\pi b = 2\pi^2 ab^2$

Example 5. *The loop of the curve* $x(x^2 + y^2) = a(x^2 + y^2)$ *revolves about the straight line* $y = 2a$. *Find the volume of the solid generated.*

Solution. The equatio of the curve can be written as

$$y^2(x + a) = x^2(a - x)$$

or

$$y^2 = \frac{x^2(a - x)}{a + x}$$

The curve is symmetrical about the X-axis and cuts the X-axis at O(0, 0) and A(a, 0)

Area of the loop $= 2\int_0^a y\,dx$

$$= 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx = 2\int_0^a \frac{x(a-x)}{\sqrt{a^2 - x^2}}\,dx$$

Put $x = a \sin\theta$, then $dx = a\cos\theta\, d\theta$

when $x = 0$, $\theta = 0$; when $x = a$, $\theta = \frac{\pi}{2}$

$\therefore$ Area of the loop $= 2\int_0^{\pi/2} \dfrac{a \sin\theta\,(a - a\sin\theta)}{a\cos\theta}\, a\cos\theta\, d\theta$

$$= 2a^2\int_0^{\pi/2} (\sin\theta - \sin^2\theta)\, d\theta = 2a^2\left(1 - \frac{\pi}{4}\right) = \frac{a^2}{2}(4-\pi)$$

The loop being symmeterical about the X-axis, the C.G. of the loop will be open the X-axis and the distance of the C.G. from $y = 2a$ (the axis of rotation) is $2a$.

$\therefore$ Distance moved by the C.G from $y = 2a$ (the axis of rotation) is $2a$

Required volume $= \dfrac{a^2}{4}(4-\pi)\times 4\pi a = 2\pi a^3(4-\pi)$

Example 6. *The loop of the curve $2ay^2 = x(x - a)^2$ revolves about the line $y = a$. Find the volume of the solid generated.*

Sol. The equation of the curve is $y^2 = \dfrac{x(x-a)^2}{2a}$

It is symmetrical about x-axis

It passes through the origin (0, 0)

The tangent at origin is $x = 0$

The loop is traced as x increases from 0 to a.

Area of loop $= 2\int_0^a y\,dx$

$$= 2\int_0^a \frac{\sqrt{x}(x-a)}{\sqrt{2a}}\,dx = \sqrt{\frac{2}{a}}\int_0^a (x^{3/2} - ax^{1/2})dx$$

$$= \sqrt{\frac{2}{a}}\left[\frac{2}{5}x^{5/2} - \frac{2}{3}ax^{3/2}\right]_0^a = \sqrt{\frac{2}{a}}\left[\frac{2a^2\sqrt{2}}{5} - \frac{2a^2\sqrt{a}}{3}\right]_0^a$$

$$= \sqrt{2}\cdot\frac{-4a^2}{15} = \frac{4\sqrt{2}}{15}a^2 \text{ (im magnitude)}$$

Since the loop is symmetrical about X-axis, its C.G. lies X-axis and the distance C.G. of loop from the axis revolution is a.

Length of the path traced out by C.G $= 2\pi a$

$\therefore$ Required volume $= \dfrac{4\sqrt{2}}{15}a^2 \times 2\pi a \dfrac{8\pi a^3\sqrt{2}}{15}$

Example 7. *A triangle is formed by the lines $3y = 4x$, $y = 0$ and the perpendicular on the first line from the point (5, 0). Calculate the volume obtained by revolving the triangle about the y-axis.*

Sol. The triangle is shown in the figure.

Equation of OB is $4x - 3y = 0$ ...(*i*)

Any line $\perp$ to its $3x + 4y + k = 0$

If it passes through A(5, 0)

$$k = -15$$

$\therefore$ Equation of AB is

$$3x + 4y - 15 = 0 \qquad \text{...}(ii)$$

Solving (*i*) and (*ii*), the co-ordinates of B are $\left(\frac{9}{5}, \frac{12}{5}\right)$

Also co-ordinates of O are (0, 0)

NOTES

$\therefore$ Co-ordinates of G, the C.G. of ΔOAB are $\left(\dfrac{0+5+\frac{9}{5}}{3}, \dfrac{0+0+\frac{12}{5}}{3}\right)$

i.e., $\left(\frac{34}{15}, \frac{4}{5}\right)$

Distance of G from the axis of revolution (*i.e.*, Y-axis) $= \frac{34}{15}$

Distance moved by G in one revolution $= 2\pi \times \frac{34}{15} = \frac{68\pi}{15}$

Area of ΔOAB $= \frac{1}{2}(12) = 6$

$\therefore$ Requried volume $= 6 \times \frac{68\pi}{15} = \frac{136\pi}{5}$

Example 8. *The length of an arch of the cycloid*

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

is 8a, and the area generated by revolving around the x-axis is $\frac{64\pi a^2}{3}$. *Use Pappus Theorem to find the area generated by revolving the arch about the tangent at the highest point.*

Sol. Length of arch OAB = $8a$

Let the distance MG of the centroid G of the arch OAB from x-axis = $\bar{y}$

Distance moved by the C.G in one revolution = $2\pi\bar{y}$

$\therefore$ Area of the surface generated by revolving the arch about x-axis

$$= 8a \times 2\pi\bar{y} = 16a\pi\bar{y}$$

But it is given to be $\frac{64\pi a^2}{3}$

$\therefore$ $16\pi\bar{y} = \frac{64\pi a^2}{3}$ or $\bar{y} = \frac{4a}{3}$

$\therefore$ Distance of G from the tangent at A (the highest point)

$$= \text{MA} - \text{MG} = 2a - \frac{4a}{3} = \frac{2a}{3}$$

$\therefore$ Distance moved by G in one revolution about $y = 2a$ is $2\pi \times \frac{2a}{3} = \frac{4a\pi}{3}$

$\therefore$ Area of the surface generated by revolving the arch about the tangent at

$$\text{A} = 8a \times \frac{4a\pi}{3} \cdot \frac{32\pi a^2}{3}$$

EXERCISE 8.3

1. Examine the convergent of the improper integrals:

(*i*) $\int_0^{\infty} e^{2x}\,dx$ (*ii*) $\int_0^{\infty} \frac{dx}{(1+x)^{2/3}}$ (*iii*) $\int_1^{\infty} \frac{x}{(1+x)^3}\,dx$

(*iv*) $\int_{-\infty}^{0} \cosh x\,dx$ (*v*) $\int_{-\infty}^{1} \frac{dx}{1+x^2}$ (*vi*) $\int_1^{2} \frac{dx}{\sqrt{x-1}}$

(*vii*) $\int_1^{2} \frac{dx}{x\sqrt{x^2-1}}$ (*viii*) $\int_2^{3} \frac{x-1}{\sqrt{x-2}}\,dx$ (*ix*) $\int_0^{e} \frac{dx}{x(\log x)^3}$

(*x*) $\int_1^{2} \frac{dx}{2-x}$ (*xi*) $\int_0^{1} \frac{dx}{x^2-1}$ (*xii*) $\int_0^{2a} \frac{dx}{(x-a)^2}$

(*xiii*) $\int_0^{2} \frac{dx}{2x-x^2}$

2. Express the following as Beta functions:

(*i*) $\int_0^1 x^3(1-x^2)^{3/2}\,dx$ (*ii*) $\int_0^2 x^3(8-x^3)^{-1/3}\,dx$

(*iii*) $\int_0^1 x^{m-1}(1-x^2)^{n-1}\,dx$

3. Prove that $\int_0^a (a-x)^{m-1}\cdot x^{n-1}\,dx = a^{m+n-1}\,\mathrm{B}(m,n)$

4. Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\mathrm{B}(m,n)}{a^n(1+a)^m}$

5. Express the following integrals in terms of Gamma functions:

(*i*) $\int_0^1 x^{p-1}(1-x^2)^{q-1}\,dx$ where $p > 0$, $q > 0$

(*ii*) $\int_0^a x^{p-1}(a-x)^{q-1}\,dx$ where $p > 0$, $q > 0$

6. Prove that $\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}$

7. Prove that

(*i*) $\mathrm{B}(p, q)\,\mathrm{B}(p+q, r) = \mathrm{B}(q+r, p) = \mathrm{B}(r, p), \mathrm{B}(r+p, q)$

(*ii*) $\mathrm{B}(p, q)\,\mathrm{B}(p+q, r)\,\mathrm{B}(p+q+r, s) = \frac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)\,\Gamma(s)}{\Gamma(p+q+r+s)}$

8. Evaluate $\int_0^{\infty} e^{-4x}.\,x^{3/2}\,dx$

9. The ellipse $\frac{x^2}{36}+\frac{y^2}{4}=1$ is revolved about the line $y = 2$. Find the volume of the solid generated.

[**Hint.** The line $y = 2$ is the tangent at the upper end of the mirror axis].

10. The loop of the curve $4y^2 = x(x-2)^2$ revolves about the line $y = 2$. Find the volume of the solid generated.

Answers

1. (*i*) Divergent (*ii*) Divergent

(*iii*) Converges to $\frac{3}{8}$ (*iv*) Diverges to ∞

NOTES

(*v*) Converges to $\frac{3\pi}{4}$ (*vi*) Converges to 2

(*vii*) Convergent (*viii*) Converges to $\frac{8}{3}$

(*ix*) Converges to $-\frac{1}{2}$ (*x*) Diverges to ∞

(*xi*) Divergest to $-\infty$ (*xii*) Diverges to ∞

2. (*i*) $B\left(4, \frac{5}{2}\right)$ (*ii*) $\frac{8}{3}B\left(\frac{4}{3}, \frac{2}{3}\right)$

(*iii*) $\frac{8}{3}B\left(\frac{4}{3}, \frac{2}{3}\right)$

5. (*i*) $\dfrac{\Gamma\left(\frac{p}{2}\right)\Gamma(q)}{2\Gamma\left(\frac{p}{2}+q\right)}$ (*ii*) $a^{p+q-1}\cdot\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

8. $\frac{3\sqrt{\pi}}{128}$ **9.** $48\pi^2$

10. $\frac{64\sqrt{2}}{15}\pi$

9. DOUBLE AND TRIPLE INTEGRALS, CHANGE OF ORDER

STRUCTURE

Evaluation of a Double Integral
Change of Variables in a Double Integral
Triple Integrals
Change of Variables in a Triple Integral
Change the Order of Integration

EVALUATION OF A DOUBLE INTEGRAL

The **double Integral** $\int_{\alpha}^{\beta}\left(\int_{a}^{b} f(x, y)\, dy^*\right) dx$ is obtained by integrating $f(x, y)$ over $[a, b]$ treating it as a function of y **(regarding x as a constant)** and then integrating the resulting function of x over the interval $[\alpha, \beta]$.

Similarly, by integrating $f(x, y)$ w.r.t. x first (**treating y as constant**) and y later, we can define another repeated integral

$$\int_{a}^{b}\left(\int_{\alpha}^{\beta} f(x, y)\, dx\right) dy.$$

Note 1. If the limits of integration a, b ; α, β are constants, then the order of integration is immaterial, provided the limits of integration are changed accordingly.

i.e.,
$$\int_{\alpha}^{\beta}\int_{a}^{b} f(x, y)\, dy\, dx = \int_{a}^{b}\int_{\alpha}^{\beta} f(x, y)\, dx\, dy$$

Note 2. In the case of variable limits of integration, we integrate first w.r.t. the variable having variable limits and then w.r.t. the variable with constant limits.

Example 1. *Evaluate* $\int_{0}^{3}\int_{1}^{2} xy\,(1 + x + y)\, dy\, dx$.

Sol. $\int_{0}^{3}\int_{1}^{2} xy\,(1 + x + y)\, dy\, dx = \int_{0}^{3}\left(\int_{1}^{2} (xy + x^2 y + xy^2)\, dy\right) dx$ (By Art. 2. above)

*$dy\, dx$ indicates that we are to first integrate w.r.t. y and then w.r.t. x.

NOTES

Integrating partially w.r.t. y treating x as constant,

$$= \int_0^3 \left[x\int_1^2 y\,dy + x^2\int_1^2 y\,dy + x\int_1^2 y^2\,dy \right] dx$$

$$= \int_0^3 \left[x\left(\frac{y^2}{2}\right)_1^2 + x^2\left(\frac{y^2}{2}\right)_1^2 + x\left(\frac{y^3}{3}\right)_1^2 \right] dx$$

$$= \int_0^3 \left[x\left(\frac{4}{2}-\frac{1}{2}\right) + x^2\left(\frac{4}{2}-\frac{1}{2}\right) + x\left(\frac{8}{3}-\frac{1}{3}\right) \right] dx$$

$$= \int_0^3 \left(\frac{3}{2}x + \frac{3}{2}x^2 + \frac{7}{3}x \right) dx$$

$$= \int_0^3 \left(\frac{23}{6}x + \frac{3}{2}x^2 \right) dx = \left(\frac{23}{6}\frac{x^2}{2} + \frac{3}{2}\frac{x^3}{3} \right)_0^3$$

$$= \frac{23}{12}(9) + \frac{1}{2}(27) = \frac{69}{4} + \frac{27}{2} = \frac{69+54}{4} = \frac{123}{4}.$$

Example 2. *Evaluate the double integral*

$$\int_0^{1/2}\int_0^{1} f(x, y)\,dy\,dx$$

where $$f(x,y) = \frac{x}{\sqrt{1-x^2y^2}}.$$

Sol. $$\int_0^{1/2}\int_0^1 f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^1 \frac{x}{\sqrt{1-x^2y^2}}\,dy\,dx = \int_0^{1/2}\left(\int_0^1 \frac{x}{x\sqrt{\left(\frac{1}{x}\right)^2 - y^2}}\,dy \right) dx$$

(We are first integrating w.r.t. y treating x as a constant)

$$= \int_0^{1/2}\left(\int_0^1 \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2 - y^2}}\,dy \right) dx = \int_0^{1/2} \left[\sin^{-1}\frac{y}{1/x} \right]_0^1 dx$$

$$\left(\because \int \frac{1}{\sqrt{a^2-y^2}}\,dy = \sin^{-1}\frac{y}{a} \right)$$

$$= \int_0^{1/2} (\sin^{-1} x - \sin^{-1} 0)\,dx = \int_0^{1/2} \sin^{-1} x\,dx = \int_0^{1/2} \sin^{-1} x \,.\, 1\,dx$$

Applying product rule of integration,

$$= \left[(\sin^{-1} x)x \right]_0^{1/2} - \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,x\,dx$$

$$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \frac{1}{2}\int_0^{1/2} (1-x^2)^{-1/2}\,(-2x)\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right) + \frac{1}{2}\left[\frac{(1-x^2)^{1/2}}{1/2}\right]_0^{1/2} \qquad \Big| \because \int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} \text{ if } n \neq -1.$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Example 3. *Evaluate* $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$.

Sol. $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_0^1 \left(\int_0^{x^2} e^{y/x}\, dy\right) dx = \int_0^1 \left(\frac{e^{y/x}}{1/x}\right)_0^{x^2} dx = \int_0^1 x\,(e^x - 1)\, dx$

Integrating by parts,

$$= \Big[x(e^x - x)\Big]_0^1 - \int_0^1 (e^x - x)\, dx = e - 1 - \left(e^x - \frac{x^2}{2}\right)_0^1 = e - 1 - \left(e - \frac{1}{2} - 1\right) = \frac{1}{2}.$$

Example 4. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy$.

Sol. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2 - y^2) - x^2}\, dx\, dy$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{A^2 - x^2}\, dx\, dy \quad \text{where} \quad A^2 = a^2 - y^2$$

$$= \int_0^a \left[\frac{\mathbf{x}}{\mathbf{2}}\sqrt{\mathbf{A^2 - x^2}} + \frac{\mathbf{A^2}}{\mathbf{2}} \sin^{-1}\frac{\mathbf{x}}{\mathbf{A}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2 - y^2 - x^2} + \frac{a^2 - y^2}{2}\sin^{-1}\frac{x}{\sqrt{a^2 - y^2}}\right]_0^{\sqrt{a^2 - y^2}} dy$$

$$= \int_0^a \left[\frac{\sqrt{a^2 - y^2}}{2}\sqrt{a^2 - y^2 - (a^2 - y^2)} + \left(\frac{a^2 - y^2}{2}\right)\sin^{-1}\frac{\sqrt{a^2 - y^2}}{\sqrt{a^2 - y^2}} - 0\right] dy$$

$$= \int_0^a \left(\left(\frac{a^2 - y^2}{2}\right)\sin^{-1} 1\right) dy = \int_0^a \left(\frac{a^2 - y^2}{2}\right)\frac{\pi}{2}\, dy \qquad \Big| \because \sin\frac{\pi}{2} = 1 \therefore \sin^{-1} 1 = \frac{\pi}{2}$$

$$= \frac{\pi}{4}\int_0^a (a^2 - y^2)\, dy = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi}{4} \cdot \frac{2a^3}{3} = \frac{\pi a^3}{6}.$$

Note. 1. If $f(x)$ is an **even** function of x *i.e.*, $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx.$$

2. If $f(x)$ is an **odd** function of x *i.e.*, $f(-x) = -f(x)$,

then $$\int_{-a}^{a} f(x)\, dx = 0.$$

Example 5. *Evaluate* $\iint x^2 y^2\, dx\, dy$ *over the circle* $x^2 + y^2 \le 1$.

Sol. $\because \quad x^2 + y^2 \le 1$

$\therefore \quad x^2 \le 1 \quad$ and $\quad y^2 \le 1 - x^2$

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or $\qquad x^2 \leq 1^2 \qquad$ and $\qquad y^2 \leq \sqrt{(1-x^2)^2}$

$\therefore \qquad -1 \leq x \leq 1 \qquad$ and $\qquad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

$(\because \ \mathbf{x^2 \leq a^2 \ \Rightarrow \ -a \leq x \leq a})$

$\therefore \quad \iint x^2y^2 \, dx\, dy$ over the circle $x^2 + y^2 \leq 1$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2 \, dy\right) dx \qquad \ldots(1)$$

Let us first evaluate $\displaystyle\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2 \, dy = x^2\left[\frac{y^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$

$$= \frac{x^2}{3}\left[(\sqrt{1-x^2})^3 - (-\sqrt{1-x^2})^3\right] = \frac{x^2}{3}\left[(1-x^2)^{3/2} + (1-x^2)^{3/2}\right]$$

$$= \frac{x^2}{3}\left[2(1-x^2)^{3/2}\right] = \frac{2}{3}x^2(1-x^2)^{3/2}$$

$\therefore$ (1) becomes $\displaystyle\iint x^2y^2 \, dx\, dy = \int_{-1}^{1}\frac{2}{3}x^2(1-x^2)^{3/2}\, dx = \frac{4}{3}\int_0^1 x^2(1-x^2)^{3/2}\, dx$

[$\because$ $f(x) = x^2(1-x^2)^{3/2}$ is an even function of x and

$\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ because $f(x)$ is an even function of x]

Put $\qquad x = \sin\theta$

$\therefore \qquad \dfrac{dx}{d\theta} = \cos\theta \qquad$ or $\qquad dx = \cos\theta \, d\theta$

When $\quad x = 0, \quad \sin\theta = 0, \quad$ *i.e.*, $\quad \theta = 0$

When $\quad x = 1, \quad \sin\theta = 1, \quad$ *i.e.*, $\quad \theta = \pi/2$

$$\therefore \quad \iint x^2y^2 dx\, dy = \frac{4}{3}\int_0^1 x^2(1-x^2)^{3/2}\, dx$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2\theta\,(1-\sin^2\theta)^{3/2}\cos\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta = \frac{4}{3}\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi}{24}.$$

Example 6. *Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.*

Sol. The curves are $y^2 = 4ax \qquad \ldots(1)$

and $\qquad x^2 = 4ay \qquad \ldots(2)$

Let us find points of intersection of (1) and (2),

From (2), $\qquad y = \dfrac{x^2}{4a}$

Putting in (1), $\qquad \dfrac{x^4}{16a^2} = 4ax$

or $\qquad x^4 = 64a^3x \qquad$ or $\qquad x(x^3 - 64a^3) = 0$

$\therefore$ Either $\quad x = 0 \quad$ or $\quad x^3 - 64a^3 = 0$ *i.e.*, $x = 0$ or $x^3 = 64a^3$

i.e., $\qquad x = 0 \qquad$ and $\qquad x = 4a$

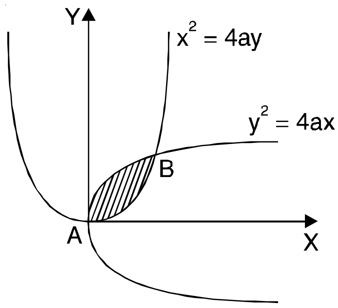
The two parabolas are $x^2 = 4ay$ and $y^2 = 4ax$

i.e., $y = \dfrac{x^2}{4a}$ and $y = 2\sqrt{a}\sqrt{x}$.

If the region between the parabolas is denoted by D, then

$$D = \left\{(x, y) : 0 \le x \le 4a, \frac{x^2}{4a} \le y \le 2\sqrt{a}\sqrt{x}\right\}$$

$\therefore$ **Area** $= \displaystyle\int_D 1\, dx\, dy$ [By Note, Art. 1.]



$$= \int_0^{4a} \left(\int_{x^2/4a}^{2\sqrt{a}\sqrt{x}} 1\, dy \right) dx$$

$$= \int_0^{4a} \Big[y \Big]_{x^2/4a}^{2\sqrt{a}\sqrt{x}} dx$$

$$= \int_0^{4a} \left[2\sqrt{a}\sqrt{x} - \frac{x^2}{4a} \right] dx$$

$$= \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a} \right]_0^{4a} = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Caution. On solving (1) and (2) for x, we have got $x = 0$ and $x = 4a$.

Putting these values in x, we could get $y = 0$ and $y = 4a$.

But we should not take the limits of integration as $0 \le x \le 4a$, $0 \le y \le 4a$ because otherwise $D = \{(x, y) : 0 \le x \le 4a, 0 \le y \le 4a\}$ would be a rectangle.

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

We have been evaluating ordinary complicated integrals by means of substitutions *i.e.*, by means of change of variable.

Exactly similarly, we can perform change of variables in double integrals.

If the variables x, y in a double integral $\displaystyle\int_R\int f(x, y)\, dx\, dy$ be changed to u, v by means of the relations $x = \phi(u, v)$; $y = \psi(u, v)$, then the double integral is transformed to

$$\int_{R'}\int f\{\phi(u, v), \psi(u, v)\}\, |J|\, du\, dv \quad \text{where}$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

and R′ is the region in $u - v$ plane corresponding to the region R of $x - y$ plane. The determinant J is called Jacobian of x, y w.r.t. u, v.

Thus, **the rule for change of variables in a double integral is :** Replace x, y by their values in terms of u, v ; the element of area $dx\, dy$ by $|J|\, du\, dv$, and the region of integration R by R′ (the corresponding region in the u, v plane).

Remark. In problems where the region is $x^2 + y^2 \le a^2$ and preferably the integrand is a function of $(x^2 + y^2)$; we use the transformations $\mathbf{x = r \cos \theta, y = r \sin \theta}$.

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Example 7. *Evaluate* $\iint x^2y^2 \, dx\, dy$ *over the circle* $x^2 + y^2 \le 1$.

Sol. Let us use the transformations

$$x = r \cos \theta,\ y = r \sin \theta$$

(*i.e.*, $x^2 + y^2 = r^2$) (*See Remark Art. 3*)

$\therefore$ The region $\{(x, y) : x^2 + y^2 \le 1\}$ is mapped into the region

$$\{ (r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi \}$$

$(x^2 + y^2 \le 1$ *i.e.*, $r^2 \le 1$. But $r \ge 0$ $\therefore$ $0 \le r \le 1)$

$$\text{Jacobian } \mathbf{J} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$\therefore$ $|\,J\,| = |\,r\,| = r$; $dx\,dy = |\,J\,|\,dr\,d\theta = r\,dr\,d\theta$

$$\therefore \iint_{x^2+y^2 \le 1} x^2y^2\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2\theta\,.\,r^2\sin^2\theta\,.\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\sin^2\theta \int_0^1 r^5\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\sin^2\theta\left[\frac{r^6}{6}\right]_0^1 d\theta$$

$$= \frac{1}{6}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta = \frac{2}{6}\int_0^{\pi}\cos^2\theta\sin^2\theta\,d\theta$$

Here $f(\theta) = \cos^2\theta\sin^2\theta$ and $f(2\pi - \theta) = f(\theta)$ $\therefore$ $\left.\int_0^{2a} f(\theta)\,d\theta = 2\int_0^a f(\theta)\,d\theta\right]$

$$= \frac{4}{6}\int_0^{\pi/2}\cos^2\theta\sin^2\theta\,d\theta = \frac{2}{3}\cdot\frac{1.1}{4.2}\frac{\pi}{2} = \frac{\pi}{24}$$

$\Big[\because \int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta$ ($p > 1, q > 1$ and p, q are integers)

$$= \frac{[(p-1)\times\text{go on decreasing by 2}][(q-1)\times\text{go on decreasing by 2}\left(\times\dfrac{\pi}{2}\text{ if } p,q \text{ are both even}\right)]}{(p+q)\times\text{go on decreasing by 2}}$$

Note. $\int_0^{\pi/2}\cos^n\theta\,d\theta$ or $\int_0^{\pi/2}\sin^n\theta\,d\theta$ (n is a positive integer > 1)

$$= \frac{(n-1)\times\text{go on dereasing by 2}}{n\times\text{go on decreasing by 2}}\left(\times\frac{\pi}{2}\text{ if } n \text{ is even}\right)$$

Example 8. *Evaluate* $\iint_{x^2+y^2 \le 1} \sin\pi(x^2+y^2)\,dx\,dy$.

Sol. Let us use the transformations $x = r \cos \theta$, $y = r \sin \theta$ (*i.e.*, $x^2 + y^2 = r^2$) (*see Remark Art. 3*).

$\therefore$ The region $\{(x, y) : x^2 + y^2 \le 1)\}$ is mapped into the region $\{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

$(x^2 + y^2 \le 1 \quad i.e., \quad r^2 \le 1. \quad \text{But } r \ge 0 \quad \therefore \quad 0 \le r \le 1)$

Jacobian $\mathrm{J} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

$\therefore \quad |\mathrm{J}| = |r| = r$

$$\therefore \quad \iint\limits_{x^2+y^2 \le 1} \sin\pi(x^2+y^2)\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (\sin\pi r^2)\,|\mathrm{J}|\,dr\,d\theta = \int_0^{2\pi}\left(\int_0^1 r\sin(\pi r^2)\,dr\right)d\theta \qquad \ldots(1)$$

Let us first evaluate $\int_0^1 r\sin(\pi r^2)^*\,dr$

Put $\quad r^2 = t$

Differentiating $\quad 2r = \frac{dt}{dr} \quad \therefore \quad r\,dr = \frac{dt}{2}$

when $\quad r = 0, \quad t = 0$

when $\quad r = 1, \quad t = 1$

$$\therefore \quad \int_0^1 r\sin\pi r^2\,dr = \int_0^1 \sin\pi t\,\frac{dt}{2} = \frac{1}{2}\left[-\frac{\cos\pi t}{\pi}\right]_0^1$$

$$= -\frac{1}{2\pi}[\cos\pi - \cos 0] = -\frac{1}{2\pi}(-2) = \frac{1}{\pi}$$

Putting this value in (1),

$$\iint\limits_{x^2+y^2\le 1} \sin\pi(x^2+y^2)\,dx\,dy = \int_0^{2\pi}\frac{1}{\pi}\,d\theta = \frac{1}{\pi}\left(\theta\right)_0^{2\pi} = \frac{2\pi}{\pi} = 2.$$

Example 9. *Evaluate* $\iint \sqrt{\dfrac{1 - x^2/a^2 - y^2/b^2}{1 + x^2/a^2 + y^2/b^2}}\,dx\,dy$

over the positive quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol. The region is $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, x \ge 0, y \ge 0$ (In First Quadrant)

Put $\quad \frac{x}{a} = u \quad$ and $\quad \frac{y}{b} = v$

$\therefore \quad x = au \quad$ and $\quad y = bv$

$\therefore \quad dx = a\,du \quad$ and $\quad dy = b\,dv$

$\therefore$ The region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ becomes circle $u^2 + v^2 \le 1$.

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*To evaluate any integral of the type $\int x\,f(x^2)\,dx$; put $x^2 = t$.

$$\therefore \quad \iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \\ x\ge 0,\, y\ge 0}} \sqrt{\frac{1-x^2/a^2-y^2/b^2}{1+x^2/a^2+y^2/b^2}}\, dx\, dy = \iint\limits_{\substack{u^2+v^2\le 1 \\ u\ge 0,\, v\ge 0}} \sqrt{\frac{1-u^2-v^2}{1+u^2+v^2}}\, ab\, du\, dv \qquad \ldots(1)$$

NOTES

Now put $u = r\cos\theta$, $v = r\sin\theta$

$\therefore$ $|J| = r$ and $du\, dv = |J|\, dr\, d\theta = r\, dr\, d\theta$.

New Limits are $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$ ($\because$ of positive quadrant)

$\therefore$ The given integral becomes

$$= ab\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta$$

Put $r^2 = t$ $\therefore$ $2r\, dr = dt$ or $r\, dr = \frac{dt}{2}$

when $r = 0$, $t = 0$

$r = 1$, $t = 1$

$\therefore$ Given Integral

$$= ab\int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-t}{1+t}}\, \frac{dt}{2}\, d\theta = \frac{ab}{2}\int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-t}{1+t}\times\frac{1-t}{1-t}}\, dt\, d\theta \qquad \text{(Rationalise)}$$

$$= \frac{ab}{2}\int_0^{\pi/2}\int_0^1 \frac{1-t}{\sqrt{1-t^2}}\, dt\, d\theta = \frac{ab}{2}\left[\int_0^{\pi/2}\left(\int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt - \int_0^1 \frac{t}{\sqrt{1-t^2}}\, dt\right) d\theta\right.$$

$$= \frac{ab}{2}\int_0^{\pi/2}\left[(\sin^{-1} t)\Big]_0^1 + \frac{1}{2}\int_0^1 (1-t^2)^{-1/2}(-2t)\, dt\right] d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2}\left[\sin^{-1} 1 - \sin^{-1} 0 + \frac{1}{2}\left(\frac{(1-t^2)^{1/2}}{1/2}\right)_0^1\right] d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2}\left[\frac{\pi}{2} - 0 + \left((1-t^2)^{1/2}\right)_0^1\right] d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2}\left(\frac{\pi}{2} + 0 - 1\right) d\theta = \frac{ab}{2}\left(\frac{\pi}{2} - 1\right)\left(\theta\right)_0^{\pi/2}$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - 1\right)\frac{\pi}{2} = \frac{ab}{2}\left(\frac{\pi^2}{4} - \frac{\pi}{2}\right) = ab\left(\frac{\pi^2}{8} - \frac{\pi}{4}\right).$$

TRIPLE INTEGRALS

The triple integral is defined in a manner entirely similar to the definition of double integral.

Like the double integral the triple integral is evaluated by reducing it to a repeated integral in which three successive integrations are performed.

We further explain it below :

For example in the following integral

$$\int_{z_1 = a}^{z_2 = b} \left[\int_{y_1 = \phi_1(z)}^{y_2 = \phi_2(z)} \left[\int_{x_1 = f_1(y, z)}^{x_2 = f_2(y, z)} f(x, y, z)\, dx \right] dy \right] dz \,,$$

we shall first integrate $f(x, y, z)$ w.r.t. x (treating y and z as constant) between the limits x_1 and x_2. The resulting expression (*i.e.*, the value of the above integral which is a function of y and z) is then integrated w.r.t. y treating z as constant between the limits y_1 and y_2. The resulting expression which is a function of z only is then integrated w.r.t. z within the limits z_1 and z_2. Hence the order of integration is from the innermost rectangle to the outermost rectangle.

The reader can easily observe from the above explanation that :

We first integrate w.r.t. the variable whose limits involve two variables, then w.r.t. the variable whose limits involve one variable and finally w.r.t. the variable whose limits are constants.

But if the limits of integration x_1, x_2 ; y_1, y_2 and z_1, z_2 are all constants, then the order of integration is immaterial, provided limits are changed accordingly.

Thus, $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\; dx\; dy\; dz$

$$= \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x, y, z)\, dy\, dz\, dx = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx.$$

Note. Volume = $\iiint$ 1 dx dy dz.

Example 10. *Evaluate* $\int_1^e \int_0^{\log y} \int_1^{e^x} \log z\; dz\; dx\; dy$ *.

Sol. $\int_1^e \int_0^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy = \int_1^e \left(\int_0^{\log y} \left(\int_1^{e^x} \log z\, dz \right) dx \right) dy$ ...(1)

$$\int_1^{e^x} \log z\, dz = \int_1^{e^x} \log z \,.\, 1\, dz$$

Integrating by parts $= \Big[\log z \,.\, z \Big]_1^{e^x} - \int_1^{e^x} \frac{1}{z} \,.\, z\, dz$

$$\Big| \because \int I \,.\, II\, dx = I \,.\int II - \int \left(\frac{d}{dx} I \int II \right) dx$$

$$= e^x \log e^x - \log 1 - \Big(z \Big)_1^{e^x} = xe^x - (e^x - 1) = (x - 1)\, e^x + 1.$$

$\therefore$ From (1), $\int_1^e \int_0^{\log y} \int_1^{e^x} \log z\; dz\; dx\; dy$

$$= \int_1^e \int_0^{\log y} [(x - 1)\, e^x + 1]\, dx\, dy \qquad ...(2)$$

**dz dx dy* indicates that we are to integrate firstly w.r.t. z ; then w.r.t. x and finally w.r.t. y.

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Now $\int_0^{\log y} [(x-1)e^x + 1]\,dx = \int_0^{\log y} (x-1)e^x\,dx + \int_0^{\log y} 1\,dx$

$$= \Big[(x-1)e^x\Big]_0^{\log y} - \int_0^{\log y} 1 . e^x\,dx + \Big[x\Big]_0^{\log y} \quad \text{(Integrating by parts)}$$

$$= (\log y - 1)\, e^{\log y} + 1 - \Big[e^x\Big]_0^{\log y} + \log y$$

$$= (\log y - 1)y + 1 - (e^{\log y} - 1) + \log y$$

$$= y\log y - y + 1 - y + 1 + \log y = (y+1)\log y - 2y + 2$$

$\therefore$ From (2), $\int_1^e \int_0^{\log y} \int_1^{e^x} \log z\,dz\,dx\,dy$

$$= \int_1^e [(y+1)\log y - 2y + 2]\,dy$$

$$= \int_1^e (\log y)(y+1)\,dy + \int_1^e (2-2y)\,dy$$

Integrating the first integral by parts,

$$= \left[\log y\left(\frac{y^2}{2} + y\right)\right]_1^e - \int_1^e \frac{1}{y}\left(\frac{y^2}{2} + y\right)dy + \Big[2y - y^2\Big]_1^e$$

$$= \frac{e^2}{2} + e - \int_1^e \left(\frac{y}{2} + 1\right)dy + 2e - e^2 - (2-1)$$

$$= \frac{e^2}{2} + e - \left(\frac{y^2}{4} + y\right)_1^e + 2e - e^2 - 1$$

$$= -\frac{e^2}{2} + 3e - 1 - \left(\frac{e^2}{4} + e - \left(\frac{1}{4} + 1\right)\right)$$

$$= -\frac{e^2}{2} + 3e - 1 - \frac{e^2}{4} - e + \frac{5}{4} = \frac{1}{4}(1 + 8e - 3e^2).$$

Example 11. *Evaluate* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *over the tetrahedron bounded by the co-ordinate planes and the plane* $x + y + z = 1$.

OR

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$.

Sol. Let the given region be R.

The region R is bounded by the co-ordinate planes ($x = 0, y = 0, z = 0$) and the plane $x + y + z = 1$.

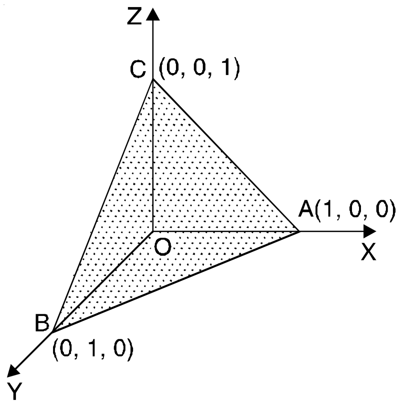
i.e., the region R is $\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$

$x + y + z \leq 1 \;\Rightarrow\; x \leq 1,\ x + y \leq 1$ and $x + y + z \leq 1$ [$\because\ x \geq 0, y \geq 0, z \geq 0$]

Now $x + y \leq 1 \;\Rightarrow\; y \leq 1 - x$

and $x + y + z \leq 1 \;\Rightarrow\; z \leq 1 - x - y$

$\therefore$ Region R is $\{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq 1 - x - y\}$



$$\therefore \quad \iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$\Bigg|\because \int x^n\,dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$$

$$= \int_0^1\int_0^{1-x}\left[\frac{1}{2(x+y+1)^2} - \frac{1}{8}\right] dy\,dx$$

$$= \int_0^1\left[\frac{-1}{2(x+y+1)} - \frac{y}{8}\right]_0^{1-x} dx$$

$$\Bigg|\because \int \frac{1}{(x+y+1)^2}\,dy = \int (x+y+1)^{-2}\,dy = \frac{(x+y+1)^{-1}}{-1} = \frac{-1}{x+y+1}$$

$$= \int_0^1\left[\frac{-1}{4} - \frac{1-x}{8} + \frac{1}{2(x+1)}\right] dx = \left[-\frac{x}{4} - \frac{(1-x)^2}{2\times 8(-1)} + \frac{1}{2}\log(x+1)\right]_0^1$$

$$= -\frac{1}{4} + \frac{1}{2}\log 2 - \frac{1}{16} = \frac{1}{2}\log 2 - \frac{5}{16}.$$

CHANGE OF VARIABLES IN A TRIPLE INTEGRAL

(*a*) Change to cylindrical polar co-ordinates

The relations between the cartesian and cylindrical polar co-ordinates of a point are given by the equations

$$\mathbf{x = r\cos\theta,\ y = r\sin\theta,\ z = z}$$

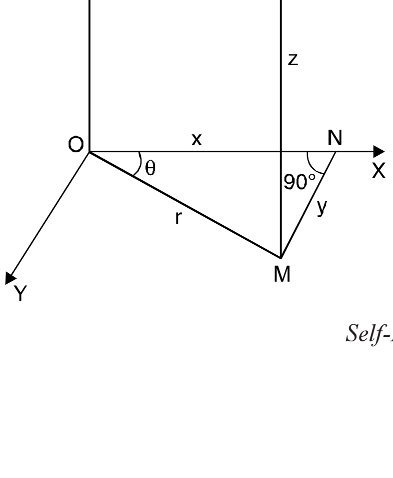
$$\therefore \qquad J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding by third column

$$= (1)\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$\therefore \quad |\,J\,| = r.$

$\therefore$ **dx dy dz is to be replaced by | J | dr dθ dz = r dr dθ dz.**



Note. Cylindrical polar co-ordinates are useful when the region of integration is a right circular cylinder.

NOTES

Example 12. *Evaluate* $\iiint\limits_{\substack{x^2+y^2\le 1\\ 2\le z\le 3}} z(x^2+y^2)\,dx\,dy\,dz$.

Sol. Let V = {(x, y, z) : $2 \le z \le 3$, $x^2 + y^2 \le 1$}.

Let us use the transformations,

x = r cos θ, y = r sin θ, z = z (cylindrical polar co-ordinates)

∴ The region of integration V′ (in terms of r, θ, z) is given by

$$V' = \{(r, \theta, z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 2 \le z \le 3\}$$

$$|\,J\,| = r \qquad \text{(see Art. 5 (a))}$$

$$\therefore \quad \iiint\limits_{\substack{x^2+y^2\le 1\\ 2\le z\le 3}} z(x^2+y^2)\,dx\,dy\,dz = \int_0^1\left(\int_0^{2\pi}\left(\int_2^3 z\,.\,r^2\,.\,r\,dz\right)d\theta\right)dr$$

[Replacing $dx\,dy\,dz$ by $|\,J\,|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$]

$$= \int_0^1\int_0^{2\pi} r^3\left[\frac{z^2}{2}\right]_2^3 d\theta\,dr = \int_0^1\left(\int_0^{2\pi}\frac{r^3}{2}\,.\,5\,d\theta\right)dr = \int_0^1 \frac{5r^3}{2}\left(\theta\right)_0^{2\pi} dr$$

$$= \int_0^1 5\pi\,r^3\,dr = 5\pi\left(\frac{r^4}{4}\right)_0^1 = 5\pi\left(\frac{1}{4}-0\right) = \frac{5\pi}{4}.$$

(*b*) **Change to Spherical Polar Co-ordinates**

The relations between the cartesian and spherical polar co-ordinates of a point are given by equations

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Take r common from C_2 and $r\sin\theta$ from C_3

$$= r\,.\,r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

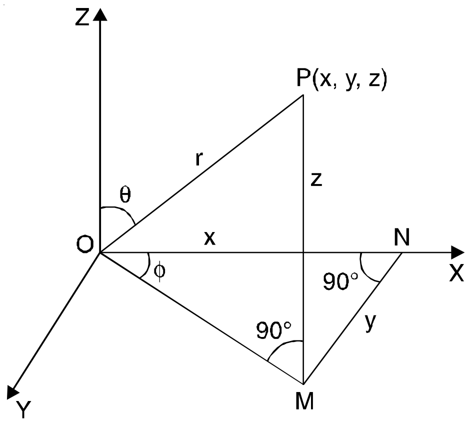
Expanding by third row

$$= r^2\sin\theta\left[\cos\theta\begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta\begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}\right]$$

$= r^2 \sin\theta$ (after simplification)

$\therefore$ **dx dy dz is to be replaced by**

$|J|$ dr dθ dϕ = r^2 sin θ dr dθ dϕ.



Note. Spherical polar co-ordinates are useful when the region of integration is a sphere or a part of it.

Example 13. *Show that for m > 0,*

$$\iiint_{x^2+y^2+z^2\le 1} (x^2+y^2+z^2)^m\, dx\, dy\, dz = \frac{4\pi}{2m+3}.$$

Sol. Let V = $\{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$.

Let us use the transformations,

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

(Spherical polar co-ordinates)

$\therefore \quad x^2 + y^2 + z^2 = r^2$

and $\quad |J| = r^2 \sin\theta$ [see Art. 5 (*b*)]

$\therefore$ The region of intersection V′ (in terms of r, θ, ϕ) is given by

$$V' = \{(r, \theta, \phi)\} : 0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}.$$

$$\therefore \quad \iiint_{x^2+y^2+z^2\le 1} (x^2+y^2+z^2)^m\, dx\, dy\, dz$$

$$\int_0^{2\pi}\int_0^{2}\int_0^{1} (r^2)^m r^2 \sin\theta\, dr\, d\theta\, d\phi \qquad (\because\ x^2+y^2+z^2=r^2)$$

[Replacing $dx\, dy\, dz$ by $|J|\, dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$]

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^{2m+2}\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \int_0^{1} r^{2m+2}\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \left[\frac{r^{2m+3}}{2m+3}\right]_0^1 d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta}{2m+3}\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \left[-\frac{\cos\theta}{2m+3}\right]_0^{\pi} d\phi = \int_0^{2\pi} \frac{-1}{2m+3}\,[\cos\pi - \cos 0]\, d\phi = \int_0^{2\pi} \frac{2}{2m+3}\, d\phi$$

$$= \frac{2}{2m+3}\int_0^{2\pi} 1\, d\phi = \frac{2}{2m+3}\Big[\phi\Big]_0^{2\pi} = \frac{4\pi}{2m+3}.$$

NOTES

Example 14. *Evaluate* $\displaystyle\iiint_{x^2+y^2+z^2\le 1} x^2\, dx\, dy\, dz$.

Sol. Let V = $\{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$.

Let us use the transformations

$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$

(Spherical Polar Co-ordinates)

$\therefore\quad x^2 + y^2 + z^2 = r^2$ and $|\text{J}| = r^2 \sin\theta$ (See Art. 5 (*b*))

$\therefore$ The region of integration V′ in terms of r, θ and ϕ is given by

$V' = \{(r, \theta, \phi) : 0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}$.

$$\therefore \iiint_{x^2+y^2+z^2\le 1} x^2\, dx\, dy\, dz$$

$$= \iiint r^2 \sin^2\theta \cos^2\phi\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

[Replacing $dx\, dy\, dz$ by $|\text{J}|\, dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$]

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4 \sin^3\theta \cos^2\phi\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \cos^2\phi \int_0^1 r^4\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \cos^2\phi \left[\frac{r^5}{5}\right]_0^1 d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \cos^2\phi \,.\, \frac{1}{5}\, d\theta\, d\phi = \frac{1}{5}\int_0^{2\pi} \cos^2\phi \int_0^{\pi} \sin^3\theta\, d\theta\, d\phi$$

$$= \frac{1}{5}\int_0^{2\pi} \cos^2\phi \,.\, 2\int_0^{\pi/2} \sin^3\theta\, d\theta\, d\phi$$

$$\left[\because \text{ If } f(x) = f(a-x), \text{ then } \int_0^a f(x)\, dx = 2\int_0^{a/2} f(x)\, dx\right]$$

$$= \frac{2}{5}\int_0^{2\pi} \cos^2\phi\, \frac{2}{3.1}\, d\phi$$

$$\left[\because \int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{(n-1)\times \text{go on decreasing by } 2}{n \times \text{go on decreasing by } 2} \times \frac{\pi}{2} \text{ if } n \text{ is even}\right]$$

$$= \frac{4}{15}\int_0^{2\pi} \cos^2\phi\, d\phi = \frac{4}{15}\int_0^{2\pi} \left(\frac{1+\cos 2\phi}{2}\right) d\phi$$

$$= \frac{2}{15}\int_0^{2\pi} (1 + \cos 2\phi)\, d\phi = \frac{2}{15}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi}$$

$$= \frac{2}{15}\left[2\pi + \frac{1}{2}\sin 4\pi - 0\right] = \frac{4\pi}{15} \qquad [\because \sin 4\pi = 0]$$

EXERCISE 9.1

1. If A is the rectangle $0 \le x \le 1$, $0 \le y \le 3$, find

$$\iint_A x^2 y^3 \, dy \, dx.$$

2. (*a*) Prove that $\int_0^1 \int_0^1 (x^2 + y^2) \, dx \, dy = \frac{2}{3}$ (*b*) Evaluate $\int_0^3 \int_0^1 (x^2 + 3y^2) \, dy \, dx$

(*c*) Evaluate $\int_1^a \int_1^b \frac{dy \, dx}{xy}$.

3. Verify that $\int_1^2 \int_3^4 (xy + e^y) \, dy \, dx = \int_3^4 \int_1^2 (xy + e^y) \, dx \, dy.$

4. Evaluate (*a*) $\int_0^1 \int_0^1 \frac{dx \, dy}{\sqrt{(1 - x^2)(1 - y^2)}}$ (*b*) $\int_0^a \int_0^b (x^2 + y^2) \, dy \, dx.$

5. Evaluate the following :

(*a*) $\int_{-1}^2 \int_{x^2}^{x+2} dy \, dx$ (*b*) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy \, dx}{1 + x^2 + y^2}$

(*c*) $\int_0^1 \int_{x^2}^{x} (x^2 + 3y + 2) \, dy \, dx$ (*d*) $\int_0^{\pi/2} \int_0^{4 \sin \theta} r \, dr \, d\theta$

(*e*) $\int_0^1 \int_y^{y^2 + 1} x^2 \, y \, dx \, dy$ (*f*) $\int_0^{\pi} \int_0^{a(1 + \cos \theta)} r \, dr \, d\theta$.

6. Evaluate the following double integrals :

(*a*) $\iint_A x \, dx \, dy$ where $A = \{(x, y) : 0 \le x \le a, 0 \le y \le x\}$

[**Hint.** Here limits of y are variable. So we have to integrate first w.r.t. y and then w.r.t. x]

(*b*) $\iint_A dx \, dy$ where $A = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\sqrt{x}\}$

7. (*a*) Evaluate the double integral $\int_0^1 \int_{\sqrt{y}}^1 dx \, dy$ and sketch the region of integration.

(*b*) Evaluate the double integral $\int_0^{\sqrt{2}} \int_{-\sqrt{4 - 2y^2}}^{\sqrt{4 - 2y^2}} y \, dx \, dy$ and sketch the region of integration.

8. Evaluate $\iint x^2 y \, dx \, dy$ over the circle $x^2 + y^2 \le 1$.

9. Evaluate $\iint x^2 y^2 \, dx \, dy$ over $\{(x, y) : x \ge 0, y \ge 0 ; x^2 + y^2 \le 1\}$.

[**Hint.** Region of Integration is $\{(x, y) : 0 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\}$]

10. (*a*) Evaluate $\iint e^{2x + 3y} \, dx \, dy$ over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

(*b*) Evaluate $\iint \cos(x + y) \, dx \, dy$ over the region bounded by $x = 0$, $y = 0$, $x + y = 1$.

11. Evaluate $\int_D\int \sqrt{4x^2 - y^2}\,dx\,dy$ where D is the triangle bounded by the lines $y = 0$, $y = x$, $x = 1$.

[**Hint.** Region of Integration is $\{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$.]

12. (*a*) Evaluate $\int\int xy\,(x + y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.

[**Hint.** $y = x^2$ and $y = x$ intersect at (0, 0) and (1, 1).

$\therefore$ The region is $0 \le x \le 1$, $x^2 \le y \le x$.]

(*b*) Find the area bounded by the parabola $y^2 = x$ and the line $y = x$.

13. Compute the value of $\int_B\int y\,dx\,dy$, where B is the region in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

[**Hint.** Region B is $x \ge 0, y \ge 0$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \Rightarrow 0 \le x \le a, 0 \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$ *OR* Do by the method explained in Example 9.]

14. Evaluate $\int_B\int y\,dx\,dy$, where B is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

15. Let $A = \{(x, y) : x^2 + y^2 \le 1\}$.

Evaluate $\int_A\int (x^2 + y^2)^{7/2}\,dx\,dy$.

16. Evaluate $\int_D\int e^{-(x^2+y^2)}\,dy\,dx$ where D is the region bounded by $x^2 + y^2 = a^2$.

17. (*a*) Evaluate $\int\int xy\,(x^2 + y^2)^{3/2}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

(*b*) Evaluate $\int\int xy\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

18. (*a*) Evaluate $\int\int \sqrt{a^2 - x^2 - y^2}\,dx\,dy$ over the semi-circle $x^2 + y^2 = ax$ in the positive quadrant.

[**Hint.** Changing to polar co-ordinates the circle $x^2 + y^2 = ax$ transforms into $r^2 = ar\cos\theta$ *i.e.*, $r = a\cos\theta$.

$\therefore$ Region of Integration is $\left\{(r, \theta) : 0 \le r \le a\cos\theta, 0 \le \theta \le \frac{\pi}{2}\right\}$.]

(*b*) Evaluate $\int\int (a^2 - x^2 - y^2)\,dx\,dy$ over the circle $x^2 + y^2 = ay$ in the positive quadrant.

19. (*a*) Find the area of the region bounded by the circle $x^2 + y^2 = a^2$.

(*b*) Find the area of the region bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = 1$.

20. Find the area of the region bounded by $y^2 = 25x$ and $x^2 = 16y$ using double integral.

21. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

22. Evaluate $\int\int \sqrt{\frac{1 - x^2 - y^2}{1 + x^2 + y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

23. (*a*) Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\, dx\, dy\, dz.$

(*b*) Show that $\int_0^a\int_0^a\int_0^a (yz+zx+xy)\, dx\, dy\, dz = \frac{3}{4}a^5$.

24. Evaluate the following :

(*a*) $\int_0^1\int_{y^2}^1\int_0^{1-x} x\, dz\, dx\, dy$ (*b*) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$.

25. Evaluate the following :

(*a*) $\int_{-1}^1\int_0^z\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$ (*b*) $\int_1^3\int_{1/x}^1\int_0^{\sqrt{xy}} xyz\, dz\, dy\, dx$

(*c*) $\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\, dz\, dy\, dx$.

26. Show that the following integrals vanish :

(*a*) $\iiint\limits_{x^2+y^2+z^2\le 1} (z^5+z)\, dx\, dy\, dz$

(*b*) $\iiint xyz\, dx\, dy\, dz$ over the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

(*c*) $\iiint\limits_{x^2+y^2+z^2\le 1} (ax+by+cz)\, dx\, dy\, dz.$

27. Evaluate $\iiint xyz\, dx\, dy\, dz$ over the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$

[**Hint.** Same as Q. 25 (*c*).]

28. Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

29. (*i*) Find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; a,b,c are positive.

(*ii*) Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$.

(*iii*) Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$

30. Find the volume of the sphere $x^2+y^2+z^2\le a^2$ where $a>0$.

OR

Find the volume of the sphere of radius a.

31. (*a*) Evaluate $\iiint z^2\, dx\, dy\, dz$ over the sphere $x^2+y^2+z^2=1$.

(*b*) Show that $\iiint x^2\, dx\, dy\, dz = \frac{4\pi}{15}$ where V is the interior of the sphere $x^2+y^2+z^2=1$.

32. Evaluate $\iiint\limits_{x^2+y^2+z^2\le 1} (x^2+y^2+z^2)\,dx\,dy\,dz$.

NOTES

33. Evaluate $\iiint (lx+my+nz)^2\,dx\,dy\,dz$ over $\{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$.

34. Evaluate $\iiint z\,(x^2+y^2+z^2)\,dx\,dy\,dz$ through the volume of the cylinder $x^2 + y^2 = a^2$ intercepted by the planes $z = 0$ and $z = h$.

35. Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$, the integral being extended to the positive octant of the sphere $x^2 + y^2 + z^2 = 1$.

[**Hint.** Change to spherical polar co-ordinates.
Region of Integration is
$\{(r, \theta, \phi) : 0 \le r \le 1,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2)\}$.]

36. Evaluate $\iint_D \int (x^2+z^2)\,dx\,dy\,dz$

where $D = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$

37. Evaluate $\iiint (ax+by+cz)\,dx\,dy\,dz$ over the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1 .$$

Answers

1. $\frac{27}{4}$ **2.** (*b*) 12 (*c*) $\log a \log b$

4. (*a*) $\frac{\pi^2}{4}$ (*b*) $\frac{ab}{3}(a^2+b^2)$

5. (*a*) $\frac{9}{2}$ (*b*) $\frac{\pi}{4}\log(1+\sqrt{2})$ (*c*) $\frac{7}{12}$

(*d*) 2π (*e*) $\frac{67}{120}$ (*f*) $\frac{3}{4}\pi a^2$

6. (*a*) $\frac{a^3}{3}$ (*b*) $\frac{4}{5}$ **7.** (*a*) $\frac{1}{3}$ (*b*) $\frac{8}{3}$

8. 0 **9.** $\frac{\pi}{96}$

10. (*a*) $\frac{1}{6}(2e^3-3e^2+1)$ (*b*) $\sin 1 + \cos 1 - 1$ **11.** $\frac{1}{3}\left[\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right]$

12. (*a*) $\frac{3}{56}$ (*b*) $\frac{1}{6}$ **13.** $\frac{ab^2}{3}$ **14.** $\frac{48}{5}$

15. $\frac{2\pi}{9}$ **16.** $\pi(1-e^{-a^2})$ **17.** (*a*) $\frac{1}{14}$ (*b*) $\frac{a^4}{8}$

18. (*a*) $\dfrac{a^3}{3}\left(\dfrac{\pi}{2}-\dfrac{2}{3}\right)$ (*b*) $\dfrac{5\pi a^4}{16}$ **19.** (*a*) πa^2 (*b*) $\dfrac{\pi}{4}$

20. $\dfrac{400}{3}$ **21.** πab **22.** $\dfrac{\pi^2}{8}-\dfrac{\pi}{4}$

23. (*a*) $(e-1)^3$ **24.** (*a*) $\dfrac{4}{35}$ (*b*) $\dfrac{1}{48}$

25. (*a*) 0 (*b*) $\dfrac{13}{9}-\dfrac{1}{6}\log 3$ (*c*) $\dfrac{1}{720}$ **27.** $\dfrac{1}{720}$

28. $\dfrac{1}{8}$ **29.** (*i*) $\dfrac{abc}{6}$ (*ii*) $\dfrac{1}{6}$ (*iii*) 1

30. $\dfrac{4}{3}\pi a^3$ **31.** (*a*) $\dfrac{4\pi}{15}$ **32.** $\dfrac{4\pi}{5}$

33. $\dfrac{4\pi}{15}(l^2+m^2+n^2)$ **34.** $\dfrac{\pi}{4}a^2h^2(a^2+h^2)$ **35.** $\dfrac{\pi^2}{8}$

36. $\dfrac{4\pi}{15}abc(c^2+a^2)$ **37.** 0

CHANGE THE ORDER OF INTEGRATION

We know by Note 1, Art. 2 that if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly.

Thus, $$\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx\,.$$

But if the limits of y are functions of x, we have to find the new limits of x as functions of y while changing the order of integration.

For finding the new limits of integration to change the order of integration, a rough sketch of the region of integration is helpful.

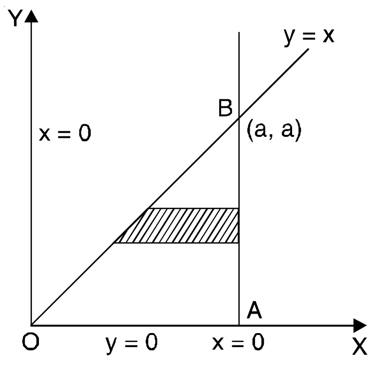
Example 1. *Change the order of integration in the integral*

$$\int_0^a\int_0^x f(x,y)\,dy\,dx\,.$$

Sol. Here, we are to integrate first w.r.t. y and then w.r.t. x.

The given limits show that the region of integration is bounded by the curves $y=0$, $y=x$, $x=0$ and $x=a$.

Let us draw a rough sketch of this region of integration.



Now $y=0$ represents x axis and $y=x$ represents a st. line through the origin.

Also $x=0$ represents y-axis and $x=a$ represents a st. line parallel to y-axis. Therefore, the region of integration is the triangle OAB in the adjoining figure and B is (a, a) [obtained by solving $y=x$ and $x=a$].

For the given order*, the region of integration is divided into vertical strips. To change the order of integration, we will first integrate w.r.t. x and then w.r.t. y.

For changing the order of integration, we divide the region into **horizontal** strips.

NOTES

The new limits of integration become :

The horizontal strips start from the line $x = y$ [from $y = x$, we have $x = y$] and end on the line $x = a$. [See the figure].

Also for this region, these strips start from $y = 0$ and end on $y = a$.

Hence on changing the order of integration, given double integral becomes

$\int_0^a \int_y^a f(x, y)\, dx\, dy$.

Example 2. *Change the order of integration in*

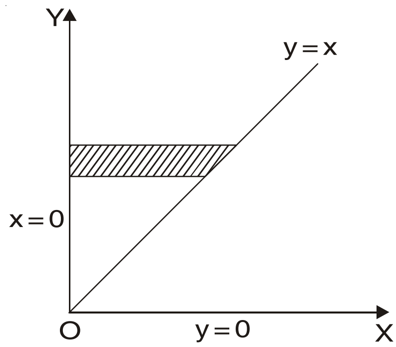
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\, dx \text{ and hence find its value.}$$

Sol. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\, dx \Rightarrow$ we are to integrate first w.r.t. y and then w.r.t. x.

The given limits show that the region of integration is bounded by the lines **y = x and x = 0.**

Let us draw a rough sketch of this region of integration.

Now $y = x$ represents a st. line passing through the origin and $x = 0$ represents y-axis.



For the given order, the region of integration is divided into **vertical** strips.

To change the order of integration, we will first integrate w.r.t. x and then w.r.t. y.

For changing the order of integration, we divide the region into **horizontal** strips. (See footnote given below)

The new limits of integration become :

The horizontal strips start from the line $x = 0$ and end on the line $x = y$. (See Figure)

(From $y = x$, we have $x = y$)

Also for this region, these strips start from $y = 0$ and extend to $y = \infty$.

Hence on changing the order of integration, the given double integral becomes

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\, dy$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y} x\right]_{x=0}^{y} dx = \int_0^\infty \left(\frac{e^{-y}}{y} y - 0\right) dy$$

$$= \int_0^\infty e^{-y}\, dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty = -\left[e^{-\infty} - e^0\right] = -(0 - 1) = 1.$$

*When the limits of y are variable, then given region is divided into **Vertical** strips and to change the order of integration, we divide the given region into **Horizontal** strips. But if the limits of x are variable, the given region is divided into **Horizontal** strips and to change the order of integration, we divide the given region into **vertical** strips.

Example 3. *Change the order of integration of* $\int_0^a \int_0^{\sqrt{2ay-y^2}} f(x,y)\,dx\,dy$ *and verify the result by taking f(x, y) = 2x.*

Sol. $\int_0^a \int_0^{\sqrt{2ay-y^2}} f(x,y)\,dx\,dy \Rightarrow$ we are to integrate first w.r.t. x and then w.r.t. y.

The given limits show that the region of integration is bounded by the curves $x = 0$, $x = \sqrt{2ay - y^2}$; $y = 0$ and $y = a$.

Let us draw a rough sketch of the region of integration.

$x = 0$ represents y-axis.

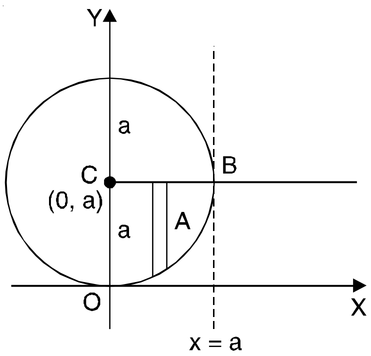
The equation $x = \sqrt{2ay - y^2}$

or squaring, $x^2 = 2ay - y^2$ ...(1)

or $x^2 + y^2 - 2ay = 0$

or $x^2 + y^2 - 2ay + a^2 = a^2$

i.e., $x^2 + (y - a)^2 = a^2$ which represents a circle whose centre is $(0, a)$ and radius a. (This circle passes through origin). $y = 0$ represents x-axis and $y = a$ represents a line || to x-axis at a distance a from it.



Therefore, the region of integration is the area OABC.

For the given order the region of integration is divided into **horizontal** strips. (See footnote page 352)

To change the order of integration we will first integrate w.r.t. y and then w.r.t. x.

For changing the order of integration, we divide the region OABC into **vertical** strips.

To find new limits of integration :

From (1), $y^2 - 2ay + x^2 = 0$

$$\therefore \quad y = \frac{2a \pm \sqrt{4a^2 - 4x^2}}{2} = a \pm \sqrt{a^2 - x^2}$$

But for the region OABC, $y \le a$

$$\therefore \quad y = a - \sqrt{a^2 - x^2}.$$

The vertical strips start (See Figure) from the circle $y = a - \sqrt{a^2 - x^2}$ and end on the line $y = a$.

Also for this region these strips start from $x = 0$ and end on $x = a$.

Hence on changing the order of integration, the given double integral becomes

$$\int_0^a \int_{a-\sqrt{a^2-x^2}}^{a} f(x,y)\,dy\,dx.$$

NOTES

Verification $f(x, y) = 2x$.

$$\int_{y=0}^{a}\int_{x=0}^{\sqrt{2ay-y^2}} f(x,y)\,dx\,dy = \int_{y=0}^{a}\left(\int_{x=0}^{\sqrt{2ay-y^2}} 2x\,dx\right)dy$$

$$= \int_0^a \left[x^2\right]_0^{\sqrt{2ay-y^2}} dy$$

$$= \int_0^a (2ay - y^2)\,dy = \left(ay^2 - \frac{y^3}{3}\right)_0^a = a^3 - \frac{a^3}{3} = \frac{2a^3}{3} \quad ...(2)$$

Again,

$$\int_0^a \int_{a-\sqrt{a^2-x^2}}^{a} f(x,y)\,dy\,dx = \int_0^a \int_{y=a-\sqrt{a^2-x^2}}^{a} 2x\,dy\,dx$$

$$= \int_0^a \left(2xy\right)_{y=a-\sqrt{a^2-x^2}}^{a} dx = \int_0^a 2x\,[a - (a - \sqrt{a^2-x^2})]\,dx$$

$$= \int_0^a 2x\sqrt{a^2-x^2}\,dx$$

$$= -\int_0^a (a^2-x^2)^{1/2}(-2x)\,dx = -\left[\frac{(a^2-x^2)^{3/2}}{3/2}\right]_0^a$$

$$\because \int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} \text{ if } n \neq -1$$

$$= -\frac{2}{3}[0 - (a^2)^{3/2}] = -\frac{2}{3}(-a^3) = \frac{2}{3}a^3 \quad ...(3)$$

From (2) and (3), we can say that

$$= \int_0^a \int_0^{\sqrt{2ay-y^2}} f(x,y)\,dx\,dy = \int_0^a \int_{a-\sqrt{a^2-x^2}}^{a} f(x,y)\,dy\,dx \text{ where } f(x, y) = 2x.$$

EXERCISE 9.2

1. Change the order of integration in $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ and hence evaluate the same.

2. Change the order of integration and evaluate $\int_0^1 \int_{4y}^{4} e^{x^2}\,dx\,dy$.

3. Change the order of integration of the integral $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ and hence evaluate it.

4. Change the order of integration in the integral $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$.

5. Change the order of integration in

$I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ and hence evaluate the same.

Answers

1. $\int_0^a \int_0^x \frac{x\,dy\,dx}{x^2 + y^2}, \frac{\pi a}{4}$

2. $\frac{1}{8}(e^{16} - 1)$

3. $\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\,dy, \frac{16a^2}{3}$

4. $\int_0^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} f(x, y)\,dy\,dx$

5. $\int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy\,; \frac{3}{8}.$

NOTES

10. DIRICHLET'S INTEGRALS

STRUCTURE

Liouville's Extension Of Dirichlet's Theorem

In this chapter we are going to discuss Dirichlet's integrals and their applications in evaluating double and triple integrals, and volumes of some solids.

Prove : $\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} a^{l+m}$**, where D is the domain** $x \ge 0$, $y \ge 0$ **and** $x + y \le a$**. Hence, establish Dirichlet's integral* :**

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

where V is the region : $x \ge 0, y \ge 0, z \ge 0$ **and** $x + y + z \le 1$**.**

Sol. Putting $x = a\mathrm{X}$ and $y = a\mathrm{Y}$, then given integral becomes

$$\mathrm{I} = \iint_{D'} (a\mathrm{X})^{l-1} (a\mathrm{Y})^{m-1} a^2\, d\mathrm{X}\, d\mathrm{Y}$$

where D′ is the domian X ≥ 0, Y ≥ 0 and X + Y ≤ 1

$$= a^{l+m} \int_0^1 \int_0^{1-\mathrm{X}} \mathrm{X}^{l-1} y^{m-1}\, d\mathrm{Y}\, d\mathrm{X}$$

$$= a^{l+m} \int_0^1 \mathrm{X}^{l-1} \left[\frac{\mathrm{Y}^m}{m}\right]_0^{1-\mathrm{X}} d\mathrm{X} = \frac{a^{l+m}}{m} \int_0^1 \mathrm{X}^{l-1} (1-\mathrm{X})^m\, d\mathrm{X}$$

$$= \frac{a^{l+m}}{m} \mathrm{B}(l, m+1) = \frac{a^{l+m}}{m} \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= a^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} \qquad (\because \ \Gamma(m+1) = m\,\Gamma(m))$$

To establish Dirichlet's integral.

*Named after a German Mathematician P.G.L. Dirichlet (1805 – 1859) and is known for his contribution in Fourier series and number theory.

Taking $y + z \leq 1 - x = a$ (say), the given integral

$$\text{I} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1}\, y^{m-1}\, z^{n-1}\, dz\, dy\, dx$$

$$= \int_0^1 x^{l-1} \left[\int_0^a \int_0^{a-y} y^{m-1}\, z^{n-1}\, dz\, dy \right] dx$$

$$= \int_0^1 x^{l-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\, a^{m+n}\, dx \text{ (from above)}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1}\, (1-x)^{m+n}\, dx, \text{ since } a = 1 - x$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\, \text{B}(l,\, m+n+1)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Generalization. *Dirichelet's integral can be generalized to n variables :*

$$\iint \cdots \int x_1^{l_1-1}\, x_2^{l_2-1} \ldots\ldots x_n^{l_n-1}\, dx_1\, dx_2 \ldots\ldots dx_n$$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2) \ldots\ldots \Gamma(l_n)}{\Gamma(l_1 + l_2 + \ldots\ldots + l_n + 1)}$$

where the integral is extended to all positive values of the variables subject to the condition $x_1 + x_2 + \ldots\ldots + x_n \leq 1$.

We give below some examples illustrating the use of Dirichlet's integral in evaluating double and triple integrals.

Example 1. *The plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also, find its mass if the density at any point is kxyz.*

Sol. Put $\frac{x}{a} = \text{X},\ \frac{y}{b} = \text{Y},\ \frac{z}{c} = \text{Z}$

Then, $\quad \text{X} \geq 0,\ \text{Y} \geq 0,\ \text{Z} \geq 0$ and $\text{X} + \text{Y} + \text{Z} \leq 1$

Also, $\quad dx = a\, d\text{X},\ dy = b\, d\text{Y},\ dz = c\, d\text{Z}.$

$\therefore$ Volume OABC

$$= \iiint_{\text{D}} dx\, dy\, dz$$

$$= \iiint_{\text{D}'} abc\, d\text{X}\, d\text{Y}\, d\text{Z}, \text{ where } \text{X} + \text{Y} + \text{Z} \leq 1$$

$$= abc \iiint_{\text{D}'} \text{X}^{1-1}\, \text{Y}^{1-1}\, \text{Z}^{1-1}\, du\, dv\, dw$$

$$= abc \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3\,!} = \frac{abc}{6} \qquad (\because\ \Gamma(1) = 1)$$

Mass

$$= \iiint_{\text{D}} (k\, xyz)\, dx\, dy\, dz$$

$$= \iiint_{\text{D}'} k(a\text{X})\, (b\text{Y})\, (c\text{Z})\, abc\, d\text{X}\, d\text{Y}\, d\text{Z}$$

$$= k\, a^2 b^2 c^2 \iiint_{D'} X^{2-1}\, Y^{2-1}\, Z^{2-1}\, du\, dv\, dw$$

NOTES

$$= k\, a^2 b^2 c^2 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} = k\, a^2 b^2 c^2 \frac{1!\,1!\,1!}{6!} = \frac{k\, a^2 b^2 c^2}{720}.$$

Example 2. (*i*) *Evaluate* $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz$, *where x, y, z are always positive but limited by the condition* $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$.

(*ii*) *Show that if l, m, n are all positive,*

$$\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{a^l\, b^m\, c^n}{8} \cdot \frac{\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{l+m+n+2}{2}\right)}$$

where the triple integral is taken throughout the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, *which lies in first octant.*

Sol. (*i*) Put $\left(\frac{x}{a}\right)^p = X,\ \left(\frac{y}{b}\right)^q = Y,\ \left(\frac{z}{c}\right)^r = Z$

$\Rightarrow$ $x = aX^{1/p},\ y = bY^{1/q},\ z = cZ^{1/r}$.

$\Rightarrow$ $dx = \frac{a}{p} X^{1/p-1}\, dX$, etc.

$\therefore$ given integral $= \iiint \frac{a^l\, b^m\, c^n}{pqr}\, X^{\left(\frac{l}{p}-1\right)} \cdot Y^{\left(\frac{m}{q}-1\right)} \cdot Z^{\left(\frac{n}{r}-1\right)}\, dX\, dY\, dZ$

where $X + Y + Z \le 1$.

$$= \frac{a^l\, b^m\, c^n}{pqr} \cdot \frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)} \quad \text{(By Dirichlet's integral)}$$

(*ii*) Proceed as in part (*i*) taking $p = q = r = 2$.

Example 3. *Using Dirichlet's integral, evaluate the volume of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Sol. Volume $V = 8 \iiint dx\, dy\, dz$. (Using symmetry)

where $x \ge 0,\ y \ge 0,\ z \ge 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.

Put $X = \frac{x^2}{a^2},\ Y = \frac{y^2}{b^2},\ Z = \frac{z^2}{c^2}$

$\Rightarrow$ $x = aX^{1/2},\ y = bY^{1/2},\ z = cZ^{1/2}$

$\Rightarrow$ $dx = \frac{1}{2} aX^{1/2-1}\, dX$, etc. and $X \ge 0,\ Y \ge 0,\ Z \ge 0,\ X + Y + Z \le 1$.

$\therefore \qquad V = 8 \iiint \frac{1}{8} abc\, X^{1/2-1} Y^{1/2-1} Z^{1/2-1}\, dX\, dY\, dZ$

$$= abc \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} \qquad \text{(Using Dirichlet's integral)}$$

$$= abc \frac{\pi\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4}{3}\pi\, abc.$$

Example 4. *Using Dirichlet's theorem, find the volume bounded by the surface*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4} = 1.$$

Sol. $\qquad V = 8 \iiint dx\, dy\, dz$

Put $\qquad X = \frac{x^2}{a^2},\ Y = \frac{y^2}{b^2},\ Z = \frac{z^4}{c^4}.$

$\Rightarrow \qquad x = aX^{1/2},\ y = bY^{1/2},\ z = cZ^{1/4}.$

$\Rightarrow \qquad dx = \frac{a}{2} X^{1/2-1}\, dX,\ dy = \frac{b}{2} Y^{1/2-1}\, dY,\ dZ = \frac{c}{4} Z^{1/4-1}\, dZ$

$\therefore \qquad V = 8 \iiint \frac{abc}{16} X^{1/2-1} Y^{1/2-1} Z^{1/4-1}\, dX\, dY\, dZ$

$$= \frac{abc}{2} \iiint X^{1/2-1} Y^{1/2-1} Z^{1/4-1}\, dX\, dY\, dZ$$

$$= \frac{abc}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+1\right)} = \frac{\frac{1}{2} abc\, \pi\, \Gamma\left(\frac{1}{4}\right)}{\frac{5}{4}\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{8\pi\, abc}{5}.$$

Example 5. *Using Dirichlet's integral, show that the volume of the solid equation whose surface is*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1,\ is\ \frac{4\pi\, abc}{35}.$$

Sol. $\qquad V = 8 \iiint dx\, dy\, dz$

Put $\qquad X = \left(\frac{x}{a}\right)^{2/3},\ Y = \left(\frac{y}{b}\right)^{2/3},\ Z = \left(\frac{z}{c}\right)^{2/3}$

$\Rightarrow \qquad x = aX^{3/2},\ y = bY^{3/2},\ z = cZ^{3/2}$

$\Rightarrow \qquad dx = \frac{3}{2} aX^{1/2}\, dX, \quad dy = \frac{3}{2} bY^{1/2}\, dY, \quad dz = \frac{3}{2} cZ^{1/2}\, dZ$

$\therefore \qquad V = 8 \iiint \frac{27}{8} abc\, X^{1/2} Y^{1/2} Z^{1/2}\, dX\, dY\, dZ$

where $X + Y + Z \le 1$

NOTES

$$= 27\, abc \iiint X^{3/2-1}\, Y^{3/2-1}\, Z^{3/2-1}\, dX\, dY\, dZ$$

$$= 27\, abc\, \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+1\right)}$$

$$= 27\, abc\, \frac{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\pi\sqrt{\pi}}{\Gamma\left(\frac{11}{2}\right)} \qquad \left(\because\ \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}\right)$$

$$= \frac{27\, abc \times \frac{1}{8}\pi\sqrt{\pi}}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{4\pi\, abc}{35}.$$

LIOUVILLE'S EXTENSION OF DIRICHLET'S THEOREM

If x, y, z are all positive such that $h_1 \le x + y + z \le h_2$, then

$$\iiint F(x+y+z)\, x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(h)\,.\, h^{l+m+n-1}\, dh.$$

Proof is beyond the scope of this book.

Note that LIOUVILLE'S extension can be extended to *n* variables.

Let us give below some examples illustrating the use of Liouville's extension of Dirichlet's theorem.

Example 6. *Evaluate* $\iiint x^{-1/2}\,.\,y^{-1/2}\,.\,z^{-1/2}\,(1-x-y-z)^{1/2}\,dx\,dy\,dz$ *extended to all positive values of the variable subject to the condition* $x + y + z \le 1$.

Sol. The given integral

$$= \iiint x^{1/2-1}\, y^{1/2-1}\, z^{1/2-1}\, F(x+y+z)\, dx\, dy\, dz$$

where $F(x + y + z) = (1 - x - y - z)^{1/2}$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)} \int_0^1 (1-h)^{1/2}\,.\,h^{1/2+1/2+1/2-1}dh \qquad (\because\ F(h) = (1 - h)^{1/2})$$

$$= \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^3}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 h^{1/2}\,(1-h)^{1/2}\,dh = \frac{\Gamma\left(\frac{1}{2}\right)^3}{\Gamma\left(\frac{3}{2}\right)} B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)^3}{\Gamma\left(\frac{3}{2}\right)}\, \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = \frac{\frac{1}{2}\pi^2}{2\,.\,1} = \frac{\pi^2}{4}.$$

Example 7. *Evaluate the double integral*

$$I = \iint x^{1/2}\, y^{1/2}\, (1 - x - y)^{2/3}\, dx\, dy$$

over the region bounded by the lines x = 0, y = 0, x + y = 1.

Sol. $$I = \iint x^{1/2}\, y^{1/2}\, (1 - x - y)^{2/3}\, dx\, dy$$

$$= \iint x^{3/2-1}\, y^{3/2-1}\, F(x + y)\, dx\, dy$$

where $F(x + y) = (1 - x - y)^{2/3}$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} \int_0^1 (1 - h)^{2/3} \,.\, h^{3/2 + 3/2 - 1}\, dh \qquad (\because \quad F(h) = (1 - h)^{2/3})$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} B\left(\frac{5}{3}, 3\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \frac{\Gamma\left(\frac{5}{3}\right)\Gamma(3)}{\Gamma\left(\frac{5}{3}+3\right)}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{\frac{11}{3}\cdot\frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\Gamma\left(\frac{2}{3}\right)} = \frac{27\pi}{1760} \qquad \left(\because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

Example 8. *Evaluate* $\iiint \log (x + y + z)\, dx\, dy\, dz$*, the integral extending over all positive values x, y, z subject to the condition x + y + z < 1.*

Sol. $$\iiint \log (x + y + z)\, dx\, dy\, dz = \iiint x^{1-1}\, y^{1-1}\, z^{1-1}\, F(x + y + z)\, dx\, dy\, dz,$$

where $F(x, y, z) = \log (x + y + z)$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1 + 1 + 1)} \int_0^1 F(h) \,.\, h^{1+1+1-1}\, dh \qquad \text{(Liouville's Theorem)}$$

$$= \frac{1.1.1}{2.1} \int_0^1 h^2 \log h\, dh \qquad \text{(Integrating by parts)}$$

$$= \frac{1}{2}\left\{\left[(\log h)\frac{h^3}{3}\right]_0^1 - \int_0^1 \frac{1}{h}\cdot\frac{h^3}{3}\, dh\right\}, \text{ etc.}$$

$$= -\frac{1}{18}$$

$$\left[\text{Here} \lim_{h \to 0} h^3 \log h = \lim_{h \to 0} \frac{\log h}{h^{-3}} = \lim_{h \to 0} \frac{\frac{1}{h}}{-3\, h^{-4}} = \lim_{h \to 0}\left(-\frac{1}{3}h^3\right) = 0\right].$$

Example 9. *Prove that* $\iiint dx\, dy\, dz\, dw = \frac{\pi^2}{32}(b^4 - a^4)$*, where the integral is for all values of the variables for which* $x^2 + y^2 + z^2 + w^2$ *lies between* a^2 *and* b^2*, (a < b).*

Sol. The given integral I is subject to the condition $a^2 < x^2 + y^2 + z^2 + w^2 < b^2$

Putting $x^2 = X$, $y^2 = Y$, $z^2 = Z$, $w^2 = W$

$\Rightarrow \qquad x = X^{1/2}, y = Y^{1/2}, z = Z^{1/2}, w = W^{1/2}$

NOTES

$\Rightarrow \qquad dx = \frac{1}{2} X^{-1/2}\, dX$, etc.

$$\therefore \qquad I = \frac{1}{16} \iiint X^{-1/2}\, Y^{-1/2}\, Z^{-1/2}\, W^{-1/2}\, dX\, dY\, dZ\, dW$$

$$= \frac{1}{16} \iiint X^{1/2-1}\, Y^{1/2-1}\, Z^{1/2-1}\, W^{1/2-1}\, dX\, dY\, dZ\, dW$$

$$= \frac{1}{16} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)} \int_{a^2}^{b^2} h^{1/2+1/2+1/2+1/2-1}\, dh$$

$$= \frac{(\sqrt{\pi})^4}{16\,\Gamma(2)} \int_{a^2}^{b^2} h\,.\, dh \qquad \text{(Liouville's theorem)}$$

$$= \frac{\pi^2}{32} (b^4 - a^4).$$

Example 10. *Evaluate* $\iiint e^{x+y+z}\, dx\, dy\, dz,\ x > 0,\ y > 0,\ z > 0$ *and* $x + y + z \le 1$.

Sol. Here, $I = \iiint e^{x+y+z}\, x^{1-1}\, y^{1-1}\, z^{1-1}\, dx\, dy\, dz,$

$F(x + y + z) = e^{x+y+z}, \quad 0 \le x + y + z \le 1$

$$\therefore \qquad I = \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 F(h)\,.\, h^{1+1+1-1} dh \qquad \text{(By Liouville's theorem)}$$

$$= \frac{1.1.1}{2.1.} \int_0^1 e^h\, h^2\, dh$$

(Integrating by lasts, taking h^2 as first function)

$$= \frac{1}{2} (e - 2).$$

EXERCISE 10.1

1. Show that the volume of the solid bounded by the coordinate planes and the surface
$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1 \text{ is } \frac{abc}{90}.$$
$\left[\textbf{Hint.} \text{ Put } X = \left(\frac{x}{a}\right)^{1/2}, Y = \left(\frac{y}{b}\right)^{1/2}, Z = \left(\frac{z}{c}\right)^{1/2}, \text{etc.}\right]$

2. Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive octant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
$$\frac{a^m\, b^n}{2n}\, B\left(\frac{m}{2}, \frac{n}{2} + 1\right).$$
$\left[\textbf{Hint.} \text{ Put } \left(\frac{x}{a}\right)^2 = X, \left(\frac{y}{b}\right)^2 = Y.\right]$

3. Using Dirichlet's integral, show that the volume of the solid whose surface is represented by the equation $\frac{x^4}{a^4} + \frac{y^4}{b^4} + \frac{z^4}{c^4} = 1$ is $\frac{(abc)\sqrt{2}}{12\pi}\left(\Gamma\left(\frac{1}{4}\right)\right)^4$.

4. Show that the mass of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point being $\rho = k\,xyz$ is $\frac{k\,a^2\,b^2\,c^2}{48}$.

5. Show that volume in the first octant determined by the surface $x^n + y^n + z^n = a^n$, $(n > 0)$ is

$$a^3\,\frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^3}{\Gamma\left(1+\frac{3}{n}\right)}.$$

6. Show that the volume enclosed by the surface $\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}+\left(\frac{z}{c}\right)^{2n}=1$, n being a positive integer is

$$\frac{2abc\left[\Gamma\left(\frac{1}{2n}\right)\right]^3}{3n^2\,\Gamma\left(\frac{3}{2n}\right)}.$$

7. Show that the value of the integral $\iiint x^{\alpha}\,y^{\beta}\,z^{\gamma}\,(1-x-y-z)^{\lambda}\,dx\,dy\,dz$ over the interior of the tetrahedron formed by the coordinate planes and the plane $x + y + z = 1$ is

$$\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+1)}.$$

8. Show that $I = \iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,(1-x-y-z)^{p-1}\,dx\,dy\,dz$, $(l, m, n, p \geq 1)$ taken over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$ is $\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l+m+n+p)}$.

9. Prove that $\iiint_D (x+y+z+1)^2\,dx\,dy\,dz = \frac{31}{60}$

where the domain is defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$.

10. Prove : $\iiint_R \sqrt{1-x^2-y^2-z^2}\;dx\,dy\,dz = \frac{\pi^2}{32}$

where R is the region interior to the sphere $x^2 + y^2 + z^2 = 1$.

NOTES